

Private states, quantum data hiding and the swapping of perfect secrecy

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(Dated: March 22, 2017)

We derive a formal connection between quantum data hiding and quantum privacy, confirming the intuition behind the construction of bound entangled states from which secret bits can be extracted. We present three main results. First, we show how to simplify the class of private states and related states via reversible local operation and one-way communication. Second, we obtain a bound on the one-way distillable entanglement of private states in terms of restricted relative entropy measures, which is tight in many cases and shows that protocols for one-way distillation of key out of states with low distillable entanglement lead to the distillation of data hiding states. Third, we consider the problem of extending the distance of quantum key distribution with help of intermediate stations. In analogy to the quantum repeater, this paradigm has been called the quantum key repeater. We show that when extending private states with one-way communication, the resulting rate is bounded by the one-way distillable entanglement. In order to swap perfect secrecy it is thus essentially optimal to use entanglement swapping.

I. INTRODUCTION

Entanglement distillation [5] is the process of producing high fidelity *maximally entangled states* from copies of a noisy entangled state ρ , using only Local Operations and Classical Communication (LOCC), between two parties Alice and Bob. The maximally entangled states can then be used for teleportation, Bell inequality violation, etc. The rate at which they can be distilled from ρ is called the *distillable entanglement* — $E_D(\rho)$. Because maximally entangled states are pure, they are in product with the environment and therefore measuring them leads to perfectly correlated and perfectly secure pairs of bits — *perfect secret bits*. It turns out that there exist mixed states, the *private states* [18], that also lead to perfectly secure bits just by measurement. While the *distillable key* — $K_D(\rho)$ — is defined as the rate at which perfect secret bits can be distilled by local operations and *public* communication, it was shown that it also equals the rate at which private states can be distilled by LOCC. Proving this equivalence allowed the authors to make the striking discovery that distillable entanglement and distillable key can be very different for the same state [18].

There exists a low-dimensional experimental realization of the separation of distillable key and distillable entanglement with photonic states [22]. In light of this, it is natural to ask how much this separation extends to general network scenarios, and in particular whether it persists if we insert a repeater station between the two parties. In [26] first examples have been produced of states that, while containing a lot of key, do not allow for distillation of significant amounts of key across the repeater station. This may be an indication that the separation between distillable key and distillable entanglement does not survive in all general network scenarios.

Here we provide a new perspective on key distillation, and thus quantum key distribution, by intimately relating private states to quantum data hiding [10, 11]. This provides a powerful tool for the study of long-distance quantum key distribution involving intermediate repeater stations, where for the first time we are able to show a close connection with entanglement distillation. In this framework [26], noisy entanglement is distributed between the individual endpoints and the intermediate repeater station and arbitrary noiseless LOCC protocols are allowed. If this setup is used to distill maximally entangled states at the endpoints then this is an idealized version of the well known quantum repeater and if it is used to distill private states it is called a quantum key repeater. We provide an upper bound on the quantum key repeater rate with one-way classical communication; as such, the bound holds also for protocols with noisy operations that can only lower the rate and thus, if anything, leave room for improvement. Our results go beyond the use of the partial transpose and thus apply to NPT states as well as PPT invariant states, which are out of reach for [26].

The paper is organized as follows. First, we simplify the class of private states, introducing what we call Bell private states. We show that these states are, for all entanglement-related purposes, equivalent to private states. Second, the simplified structure of Bell private states allows us to confirm the intuition that the separation between distillable key and distillable entanglement is due to quantum data hiding. More precisely, we show that the states with a separation are those made of a maximally entangled state subject to phase flip error, where the error information is conserved in data hiding states. Such hidden information of the error preserves the key, but prevents Alice and Bob from correcting the maximally entangled state and distill entanglement. Third, as an application to the quantum key repeater with one-way classical communication, we show that a large class of states and protocols cannot be used to distill key across a repeater station better than performing entanglement distillation and swapping.

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II. PRIVATE STATES

Whether they are used for key distillation or other purposes, maximally entangled states are one time use only. After measuring the maximally entangled state in the computational basis to extract the perfect secret bits, the post measurement state has become separable and cannot be used to generate more perfect secret bits. For general states and in particular for states with perfect secret bits, the post measurement state is called *key attacked state*; it will play an important role in our results and thus we will always provide it in pair with the original state.

Consider bipartite systems $|A_k| = |B_k|$ — the *key* systems. Set m to $m := \log_2 |A_k|$, then the maximally entangled state Φ^m and its key attacked state $\hat{\Phi}^m$ are (in the computational basis):

$$\begin{aligned}\Phi^m &:= \frac{1}{2^m} \sum_{ij=0}^{2^m-1} |ii\rangle\langle jj|_{A_k B_k} \\ \hat{\Phi}^m &= \tau_c^m := \frac{1}{2^m} \sum_{i=0}^{2^m-1} |ii\rangle\langle ii|_{A_k B_k}\end{aligned}\quad (1)$$

where τ_c^m is the uniform mixture onto the maximally correlated subspace. For $m = 1$ we say the maximally entangled state carries a unit of pure entanglement and thus in general Φ^m carries m units of pure entanglement. The steps in quantum key distribution protocols like E91 [4] (sifting, error estimation, ...) serve the purpose, among other things, of distinguishing the maximally entangled state from its key attacked state.

The private states generalize the maximally entangled states in the context of key distillation, in the sense that they are proven to be all the states that lead to perfect secret bits when measured. In addition to the key systems $A_k B_k$ to be measured, these states reside on systems $A_s B_s$ — the *shield* systems — to be shared between Alice and Bob. $A_s B_s$ holds any information correlated with the perfect secret bits in $A_k B_k$, so that the perfect secret bits will be in product with any purifying system of the private state. A *private state* γ^m with at least m perfect secret bits and its key attacked state $\hat{\gamma}^m$ are states of the form [18]:

$$\begin{aligned}\gamma^m &:= T(\Phi_{A_k B_k}^m \otimes \sigma_{A_s B_s}) T^\dagger \\ \hat{\gamma}^m &= T(\hat{\Phi}_{A_k B_k}^m \otimes \sigma_{A_s B_s}) T^\dagger.\end{aligned}\quad (2)$$

with σ an arbitrary state and T a controlled unitary $T = \sum_{ij} |ij\rangle\langle ij| \otimes U_i$, called *twisting*. Notice that if the shield systems are absent, then the only private states are the maximally entangled states. The first examples of private states with low distillable entanglement were of the special form [18, 20]:

$$\begin{aligned}\gamma^1 &= p \cdot \phi_0 \otimes \sigma_0 + (1-p) \cdot \phi_1 \otimes \sigma_1 \\ \hat{\gamma}^1 &= p \cdot \hat{\phi}_0 \otimes \sigma_0 + (1-p) \cdot \hat{\phi}_1 \otimes \sigma_1 \\ &= \tau_c^1 \otimes (p \cdot \sigma_0 + (1-p) \cdot \sigma_1)\end{aligned}\quad (3)$$

where σ_j are arbitrary orthogonal shield states — the *shields* — and ϕ_j are the Bell states with phase flips only:

$$\begin{aligned}\phi_0 &= \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\ \phi_1 &= \frac{1}{2}(|00\rangle - |11\rangle)(\langle 00| - \langle 11|)\end{aligned}$$

namely, using the Pauli Z :

$$\begin{aligned}\phi_j &= \mathcal{Z}_{B_k}^j(\Phi^1) \\ \hat{\phi}_j &= \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) = \tau_c^1\end{aligned}\quad (4)$$

where $\mathcal{Z}^j(\varrho) = Z^j \varrho Z^{-j}$ is the map that conjugates by Z^j (Z^0 is the identity and $Z^1 = Z$). The intuition behind the examples is that, if the shields are data hiding states, they should hinder the correction of the phase flip via LOCC and thus suppress the distillable entanglement.

Not every private state, even for $m = 1$, can be written as in Equation (3) — see for example the private states in [26] — however, as we show below, we can convert all private states reversibly into this form. We call these special cases Bell private states. We first need to generalize ϕ_j to higher dimension. As in Equation (4), these are defined just by applying phase flips to the maximally entangled state:

$$\begin{aligned}\phi_j &:= \mathcal{Z}_{B_k}^j(\Phi^m) = \mathcal{Z}_{A_k}^j(\Phi^m) \\ \hat{\phi}_j &= \tau_c^m.\end{aligned}\quad (5)$$

If the dimension of A_k is a power of two, then it is sufficient to think in terms of qubits: m is an integer, j is a bit-string indexing the tensor products of Pauli's, and thus ϕ_j is just a tensor product of the Bell states of Equation (4). For the purpose of the paper it is sufficient to restrict to this case. For the general case, see the Supplementary Material.

Note that Φ and ϕ_j are pure states, in particular $\Phi = |\Phi\rangle\langle\Phi|$ and $\phi_j = |\phi_j\rangle\langle\phi_j|$, with $|\phi_j\rangle = \mathbb{1} \otimes Z^j |\Phi\rangle$. These states form a basis for the maximally correlated subspace, which is the support of τ_c :

$$\mathbb{1}_c := 2^m \tau_c^m = \sum_j \phi_j = \sum_i |ii\rangle\langle ii|. \quad (6)$$

For states ρ on $A_k B_k A_s B_s$ with support only on the maximally correlated subspace of $A_k B_k$, namely satisfying

$$(\mathbb{1}_{c, A_k B_k} \otimes \mathbb{1}_{A_s B_s}) \cdot \rho \cdot (\mathbb{1}_{c, A_k B_k} \otimes \mathbb{1}_{A_s B_s}) = \rho,$$

we can then write:

$$\begin{aligned}\rho &= \sum_{\mu\nu} |\phi_\mu\rangle\langle\phi_\nu| \otimes P_{\mu\nu} \\ \hat{\rho} &= \frac{1}{2^m} \sum_j \mathcal{Z}_{B_k}^j(\rho) \\ &= \frac{1}{2^m} \sum_{\mu\nu j} |\phi_{\mu+j}\rangle\langle\phi_{\nu+j}| \otimes P_{\mu\nu}\end{aligned}\quad (7)$$

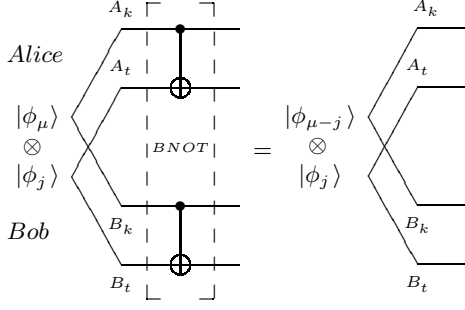


FIG. 1. Quantum circuit for the Bilateral CNOT acting on the qubit Bell states ϕ_j (dimension two), the core of the map \mathcal{E} of Construction 2.

where the $P_{\mu\nu}$ are the blocks of ρ on $A_s B_s$ in the Bell basis. We will call any such state *key correlated*. Notice how the key attacked state, obtained by measuring the computational basis, can equivalently be achieved by averaging over phase flips \mathcal{Z}^j applied uniquely on Alice's or Bob's side.

We can now define the following form of private states:

Definition 1.

A *Bell private state* is any private state of the form:

$$\begin{aligned}\gamma_{Bell}^m &= \sum_j p_j \cdot \phi_j \otimes \sigma_j \\ \hat{\gamma}_{Bell}^m &= \tau_c^m \otimes \sigma\end{aligned}\quad (8)$$

where σ_j are shields (arbitrary orthogonal states of $A_s B_s$), p_j are arbitrary probabilities and $\sigma = \sum_j p_j \sigma_j$.

The state σ is indeed the same σ as in Equation (2), it now simply has additional structure. Let us now define the reversible LOCC map that will output these Bell private states.

Construction 2. Let $A_t B_t$ be new systems — the target — with the same size as $A_k B_k$. Let V be the local unitary $V = CNOT_{A_k A_t} \otimes CNOT_{B_k B_t}$, i.e. the generalization to dimension 2^m of the Bilateral CNOT [5] illustrated in Figure 1. It holds:

$$V(|\phi_j\rangle_{A_t B_t} \otimes |\phi_\mu\rangle_{A_k B_k}) = |\phi_j\rangle_{A_t B_t} \otimes |\phi_{\mu-j}\rangle_{A_k B_k} \cdot (9)$$

Finally, define \mathcal{E} as

$$\mathcal{E}(\rho_{A_k B_k}) := V^\dagger(\tau_{c, A_t B_t}^m \otimes \rho_{A_k B_k})V$$

\mathcal{E} is one-way LOCC in either direction ($LOCC^\rightarrow$ and $LOCC^\leftarrow$, i.e. local operations with classical communication in one direction only) because τ_c^m is separable and the V is local. The map is reversible by inverting V and tracing out the target, which can be implemented with local operations only (LO).

Lemma 3. Let \mathcal{E} be the map in Construction 2. Then for any key correlated state ρ (on $A_k B_k A_s B_s$), it holds:

$$\begin{aligned}(\mathcal{E}_{A_k B_k} \otimes \text{id}_{A_s B_s})(\rho) &= \sum_j \frac{1}{2^m} \cdot \phi_j \otimes \rho_j \\ (\mathcal{E}_{A_k B_k} \otimes \text{id}_{A_s B_s})(\hat{\rho}) &= \tau_c^m \otimes \hat{\rho}\end{aligned}\quad (10)$$

where

$$\begin{aligned}\rho_j &= \mathcal{Z}_{B_k}^j(\rho) \\ \hat{\rho}_j &= \hat{\rho}.\end{aligned}\quad (11)$$

Notice that the ρ_j are locally equivalent, because they differ only by a local phase flip.

Proof. Using Equations (6), (7) and (9) we find:

$$\begin{aligned}(\mathcal{E} \otimes \text{id})(\rho) &= \sum (V^\dagger \otimes \mathbb{1})(\tau_c^m \otimes |\phi_\mu\rangle\langle\phi_\nu| \otimes P_{\mu\nu})(V \otimes \mathbb{1}) \\ &= \sum \frac{1}{2^m} (V^\dagger(\phi_j \otimes |\phi_\mu\rangle\langle\phi_\nu|)V \otimes P_{\mu\nu}) \\ &= \sum \frac{1}{2^m} (\phi_j \otimes |\phi_{\mu+j}\rangle\langle\phi_{\nu+j}| \otimes P_{\mu\nu}) \\ &= \sum \frac{1}{2^m} \cdot \phi_j \otimes \rho_j\end{aligned}$$

The key attacked state of Equation (10) follows in a similar fashion. \square

Corollary 4. Let \mathcal{E} be the map in Construction 2. Then for any private state γ^m , it holds:

$$\begin{aligned}\gamma_{Bell}^m &= (\mathcal{E}_{A_k B_k} \otimes \text{id}_{A_s B_s})(\gamma^m) = \sum_j \frac{1}{2^m} \cdot \phi_j \otimes \gamma_j^m \\ \hat{\gamma}_{Bell}^m &= (\mathcal{E}_{A_k B_k} \otimes \text{id}_{A_s B_s})(\hat{\gamma}^m) = \tau_c^m \otimes \hat{\gamma}^m\end{aligned}\quad (12)$$

where

$$\begin{aligned}\gamma_j^m &= \mathcal{Z}^j(\gamma) \\ \hat{\gamma}_j^m &= \hat{\gamma}.\end{aligned}$$

Notice that the Bell private state has the same key system size but higher dimensional shields, and the shields are themselves private states.

Proof. γ_j^m are orthogonal states because ϕ_j are orthogonal states, thus Equation (12) is indeed a Bell private state by definition. The phase flip \mathcal{Z}^j commutes with conjugation by T to produce ϕ_j . \square

In other words, anything that can be proven about entanglement monotones (entanglement measures like distillable entanglement and distillable key) for Bell private states can be proven for private states and vice versa. For example, one can always convert the output of a key distillation protocol into an approximate Bell private state, thus simplifying the expression of the distillable key to the rate at which Bell private states can be distilled.

Next, we will establish a connection between the distillable entanglement and the data hiding properties of Bell private states, then we will use Lemma 3 to generalize the result to all private states.

III. ENTANGLEMENT DISTILLATION AND QUANTUM DATA HIDING

We now show that Bell private states with low distillable entanglement are states that hide the phase of the maximally entangled states from local detection. Specifically, we give a lower bound on the one-way distillable entanglement $E_D^{\rightarrow}(\rho)$, which is the entanglement distillable with $LOCC_{A:B}^{\rightarrow}$, and consequently on the distillable entanglement. This lower bound is the rate achieved by the best entanglement distillation protocol that first performs a measurement on Alice's shield and then uses the classical information to distill.

Lemma 5. *For any key correlated state ρ that is diagonal and uniform on the Bell basis of $A_k B_k$, i.e. of the form*

$$\rho = \frac{1}{2^m} \sum_j \phi_j \otimes \sigma_j \quad (13)$$

$$\hat{\rho} = \tau_c^m \otimes \sigma$$

with $\sigma = \sum_j \frac{1}{2^m} \sigma_j$, it holds:

$$E_D^{\rightarrow}(\rho) \geq \sup_{\mathcal{M}_A} \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\sigma_j) \parallel \mathcal{M}_A(\sigma))$$

where the supremum extends over all measurements \mathcal{M}_A at Alice and $D(\rho \parallel \sigma) = \text{Tr}[\rho \log \rho - \rho \log \sigma]$ is the relative entropy.

Proof. Let Alice and Bob share a state ρ as above and let Alice perform a measurement $\mathcal{M} : A_s \rightarrow M$ on her shield:

$$\tilde{\rho} = \mathcal{M}_A(\rho) = \frac{1}{2^m} \sum_j \phi_j \otimes \mathcal{M}_A(\sigma_j)$$

where $\mathcal{M}_A \equiv \mathcal{M}_A \otimes \text{id}_B$. Alice now sends the outcome to Bob. We have $E_D^{\rightarrow}(\rho) \geq E_D^{\rightarrow}(\tilde{\rho})$. By the hashing bound [17]:

$$E_D^{\rightarrow}(\tilde{\rho}) \geq [H(B_k B_s M)_{\tilde{\rho}} - H(A_k B_k B_s M)_{\tilde{\rho}}],$$

where $H(S)_\rho := -\text{Tr}[\rho_S \log \rho_S]$ is the quantum entropy of ρ_S on system S . However, since the key systems are in a mixture of Bell states, tracing out Alice will leave Bob's key system in product with his shield and the measurement result in system M . Namely:

$$\begin{aligned} H(B_k B_s M)_{\tilde{\rho}} &= H(B_k)_{\tilde{\rho}} + H(B_s M)_{\tilde{\rho}} \\ &= H(A_k B_k)_{\tilde{\rho}} + H(B_s M)_{\tilde{\rho}}, \end{aligned} \quad (14)$$

where (14) holds because the mixture of Bell states is uniform and thus $H(B_k)_{\tilde{\rho}} = H(A_k B_k)_{\tilde{\rho}}$. We now use

$$H(X)_\alpha + H(Y)_\alpha - H(XY)_\alpha = D(\alpha_{XY} \parallel \alpha_X \otimes \alpha_Y)$$

(where α is a state on XY) and prove the claim:

$$\begin{aligned} E_D^{\rightarrow}(\rho) &\geq [H(A_k B_k)_{\tilde{\rho}} + H(B_s M)_{\tilde{\rho}} - H(A_k B_k B_s M)_{\tilde{\rho}}] \\ &= D(\frac{1}{2^m} \sum_j \phi_j \otimes \mathcal{M}_A(\sigma_j) \parallel \frac{1}{2^m} \sum_j \phi_j \otimes \mathcal{M}_A(\sigma)) \\ &= \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\sigma_j) \parallel \mathcal{M}_A(\sigma)) \end{aligned}$$

Taking the supremum over \mathcal{M}_A proves the claim. \square

Corollary 6. *For any key correlated state ρ , it holds:*

$$E_D^{\rightarrow}(\rho) \geq \sup_{\mathcal{M}_A} \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho}))$$

where the supremum is over all local measurements at Alice (ρ_j are defined in Equation (11)).

Proof. The result follows from Lemmas 3, 5 and the reversibility of Construction 2:

$$\begin{aligned} E_D^{\rightarrow}(\rho) &= E_D^{\rightarrow}((\mathcal{E} \otimes \text{id})(\rho)) \\ &= E_D^{\rightarrow}(\sum_j \frac{1}{2^m} \phi_j \otimes \rho_j) \\ &\geq \sup_{\mathcal{M}_A} \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho})). \end{aligned} \quad \square$$

These bounds generalize to the full (two-way) LOCC in the following way.

Corollary 7. *For any key correlated state ρ , it holds:*

$$E_D(\rho) \geq \sup_{\mathcal{M}_A \in LOCC_{A:B}} \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho}))$$

where $\mathcal{M}_A \in LOCC_{A:B}$ indicates that the optimization is over any LOCC protocol that ends with a measurement at Alice's.

Proof. Without loss of generality we can write the measurement in $LOCC_{A:B}$ as $\mathcal{M}' \circ \Lambda$, where \mathcal{M}' is a measurement at Alice's and Λ an $LOCC_{A:B}$ protocol. Then:

$$\begin{aligned} E_D(\rho) &= E_D((\mathcal{E} \otimes \text{id})(\rho)) \\ &= E_D(\sum_j \frac{1}{2^m} \phi_j \otimes \rho_j) \\ &\geq \sup_{\Lambda} E_D^{\rightarrow}(\sum_j \frac{1}{2^m} \phi_j \otimes \Lambda(\rho_j)) \end{aligned}$$

Finally, we use Lemma 5 and conclude the proof:

$$\begin{aligned} &\geq \sup_{\mathcal{M}'} \sup_{\Lambda} \frac{1}{2^m} \sum_j D(\mathcal{M}' \circ \Lambda(\rho_j) \parallel \mathcal{M}' \circ \Lambda(\hat{\rho})) \\ &= \sup_{\mathcal{M}_A \in LOCC_{A:B}} \frac{1}{2^m} \sum_j D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho})). \end{aligned} \quad \square$$

The relative entropy quantifies the distinguishability between states and the relative entropy of the measurement outcomes quantifies how much of this distinguishability is left when only local measurements are allowed. In the particular case of private states, the shields are orthogonal and thus they are perfectly distinguishable and j can be recovered with a global measurement. However, Lemma 5 implies that if the distillable entanglement is low, then the local distinguishability of the shields is low and j cannot be determined accurately locally: the shields are data hiding [10, 11].

In order to show that Corollary 6 is often more useful, we can provide an example of a Bell private state for which the bound of Lemma 5 is strictly suboptimal while the bound of Corollary 6 achieves equality.

Example 8. Let $|A_k| = |B_k| = |A_s|^2 = |B_s|^2 = 2^{2m}$. For the following example we need to use the whole Bell basis of two m qubits systems, we indicate these with ϕ_{ij} where now the first bit string i indexes the bit flips, while the second bitstring j is the one we have been using so far and indexes the phase flips. Then let $\phi_{0i} \otimes \phi_{0j}$ be the phase-flips-only Bell states of the key and let:

$$\begin{aligned}\gamma^{2m} &= \sum_{ij} \frac{1}{2^{2m}} \phi_{0i} \otimes \phi_{0j} \otimes \phi_{ij} \\ \hat{\gamma}^{2m} &= \tau_c^m \otimes \tau_c^m \otimes \tau^m\end{aligned}$$

where $\tau^m = \frac{1}{2^{2m}} = \sum_{ij} \frac{1}{2^{2m}} \phi_{ij}$ is the maximally mixed state of the shield. The bound of Lemma 5 computes to:

$$\sup_{\mathcal{M}_A} \frac{1}{2^{2m}} \sum_{ij} D(\mathcal{M}_A(\phi_{ij}) \parallel \mathcal{M}_A(\tau^m)) = m.$$

achieved measuring the computational basis, proven to be optimal using [16, Equation 8]. However, this state is distillable into $2m$ maximally entangled states with just a sequence of unitaries: ϕ_{ij} converts to ϕ_{ji} when changing from the computational basis to the conjugate basis on both sides (bilaterally), applying this change of basis between two applications of Equation (9) leads to

$$\gamma^{2m} \rightarrow \phi_{00} \otimes \phi_{00} \otimes \tau^m$$

thus proving that $E_D(\gamma^{2m}) = 2m$. Notice that Alice's local unitaries can be done before the measurement, while Bob's ones can be done because of unitary invariance of the relative entropy. We now compute the bound of Corollary 6, where we now need the states:

$$\begin{aligned}\gamma_{kl}^{2m} &= \sum_{ij} \frac{1}{2^{2m}} \phi_{0,i+k} \otimes \phi_{0,j+l} \otimes \phi_{ij} \\ \hat{\gamma}_{kl}^{2m} &= \tau_c^m \otimes \tau_c^m \otimes \tau^m\end{aligned}$$

for which the same distillation procedure leads to

$$\gamma_{kl}^{2m} \rightarrow \phi_{0k} \otimes \phi_{0l} \otimes \tau^m.$$

This allows to compute the bound of Corollary 6 as:

$$\begin{aligned}\sup_{\mathcal{M}} \frac{1}{2^{2m}} \sum_{kl} D(\mathcal{M}(\gamma_{kl}^{2m}) \parallel \mathcal{M}(\hat{\gamma}^{2m})) \\ &= \sup_{\mathcal{M}} \frac{1}{2^{2m}} \sum_{kl} D(\mathcal{M}(\phi_{0k} \otimes \phi_{0l} \otimes \tau^m) \parallel \mathcal{M}(\tau_c^m \otimes \tau_c^m \otimes \tau^m)) \\ &= \sup_{\mathcal{M}} \frac{1}{2^{2m}} \sum_{kl} D(\mathcal{M}(\phi_{0k} \otimes \phi_{0l}) \parallel \mathcal{M}(\tau_c^m \otimes \tau_c^m)) = 2m\end{aligned}$$

achieved measuring the conjugate basis. This is now optimal and performs strictly better than Lemma 5. \blacktriangle

We can now simplify our bounds. We find that the optimal measurement is still optimal even if we allow it to optimize independently for each ρ_j . This is an important feature because it suddenly allows for a regularization and we will be able to exploit this to give an expression for the one-way distillable entanglement of certain states.

Theorem 9. For any key correlated state ρ , it holds:

$$E_D^{\rightarrow}(\rho) \geq D_A(\rho \parallel \hat{\rho}) := \sup_{\mathcal{M}_A} D(\mathcal{M}_A(\rho) \parallel \mathcal{M}_A(\hat{\rho}))$$

$$E_D^{\rightarrow}(\rho) \geq D_A^{\infty}(\rho \parallel \hat{\rho}) := \lim_{n \rightarrow \infty} \frac{1}{n} D_A(\rho^{\otimes n} \parallel \hat{\rho}^{\otimes n})$$

where \mathcal{M}_A are arbitrary measurements (POVMs) on Alice's systems.

Proof. Recall that $\rho_j = \mathcal{Z}_{B_k}^j(\rho)$ (Equation (11)). Furthermore, it is straightforward to check that $\mathcal{Z}_{B_k}^j(\hat{\rho}) = \hat{\rho}$. Thus for any measurement \mathcal{M}_A at Alice, we have:

$$\begin{aligned}D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho})) \\ &= D(\mathcal{M}_A \otimes \text{id}_B(\rho_j) \parallel \mathcal{M}_A \otimes \text{id}_B(\hat{\rho})) \\ &= D(\mathcal{M}_A \otimes \mathcal{Z}_{B_k}^j(\rho_j) \parallel \mathcal{M}_A \otimes \mathcal{Z}_{B_k}^j(\hat{\rho})) \\ &= D(\mathcal{M}_A(\rho) \parallel \mathcal{M}_A(\hat{\rho})).\end{aligned}$$

where we used $\mathcal{M}_A \equiv \mathcal{M}_A \otimes \text{id}_B$ and in the last step we used the unitary invariance of the relative entropy. We can now rewrite Corollary 6 as:

$$\begin{aligned}E_D^{\rightarrow}(\rho) &\geq \sup_{\mathcal{M}_A} \frac{1}{2^m} \sum D(\mathcal{M}_A(\rho_j) \parallel \mathcal{M}_A(\hat{\rho})) \\ &= \sup_{\mathcal{M}_A} \frac{1}{2^m} \sum D(\mathcal{M}_A(\rho) \parallel \mathcal{M}_A(\hat{\rho})) \\ &= \sup_{\mathcal{M}_A} D(\mathcal{M}_A(\rho) \parallel \mathcal{M}_A(\hat{\rho}))\end{aligned}$$

proving the first claim. For the second claim we have:

$$E_D^{\rightarrow}(\rho) = \frac{1}{n} E_D^{\rightarrow}(\rho^{\otimes n}) \geq \frac{1}{n} D_A(\rho^{\otimes n} \parallel \hat{\rho}^{\otimes n}) \quad \forall n$$

because the distillable entanglement is already regularized and because $\rho^{\otimes n}$ is still a key correlated state. We can now end the proof taking the limit $n \rightarrow \infty$ because by using tensor product measurements we find:

$$D_A(\rho^{\otimes n+m} \parallel \hat{\rho}^{\otimes n+m}) \geq D_A(\rho^{\otimes n} \parallel \hat{\rho}^{\otimes n}) + D_A(\rho^{\otimes m} \parallel \hat{\rho}^{\otimes m}).$$

By Fekete's Lemma [1], this is enough to guarantee convergence, \square

Corollary 10. For any key correlated state ρ with separable key attacked state, it holds:

$$E_D^{\rightarrow}(\rho) = D_A^{\infty}(\rho \parallel \hat{\rho}).$$

Proof. The one-way distillable entanglement has the following upper bound for all separable states σ :

$$D_A^{\infty}(\rho \parallel \sigma) \geq E_D^{\rightarrow}(\rho)$$

where D_A^{∞} is defined as in Theorem 9. This follows directly from the results in [25], see the Supplementary Material for an explicit proof. Thus, if $\hat{\rho}$ is separable:

$$D_A^{\infty}(\rho \parallel \hat{\rho}) \geq E_D^{\rightarrow}(\rho) \geq D_A^{\infty}(\rho \parallel \hat{\rho}). \quad \square$$

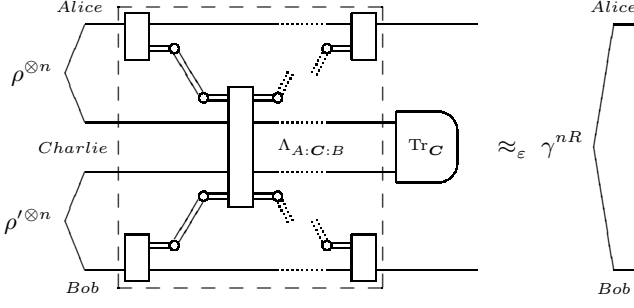


FIG. 2. Quantum circuit for the key repeater protocols in the key repeater rate $R_D(\rho, \rho')$, the rate at which private states can be distilled in a single node repeater. The dotted box is an arbitrary tripartite LOCC protocol. The double lines are the classical communication.

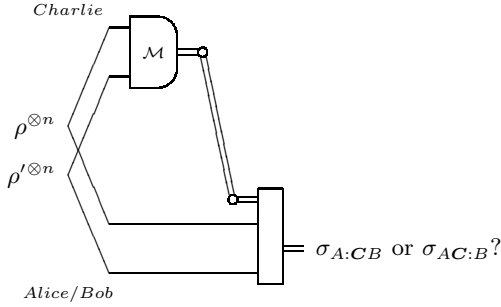


FIG. 3. Quantum circuit for the measurement in Theorem 11. This is the task of distinguishing pairs of states from separable states in $A:CB$ or $AC:B$ states by means of measurements at Charlie's. For our purposes it suffices to restrict to separable states of the form $\sigma^{\otimes n}$.

IV. QUANTUM KEY REPEATERS

We will now discuss the applications of our findings to long distance quantum communication, where noise in the communication line prevents Alice and Bob to directly share entanglement and thus secrecy, and where an intermediate repeater station, Charlie, is necessary in the mediation of the quantum correlations.

More precisely, we will assume that ρ is shared between Alice and Charlie (A and C) and ρ' is shared between Charlie and Bob (C' and B). While with a quantum repeater [8] the goal is to distill maximally entangled states between Alice and Bob, the goal with a quantum key repeater is merely to obtain perfect secret bits or, equivalently, to distill private states [26]. The rate at which this can be done will be called the quantum *key repeater rate* $R_D(\rho, \rho')$ — see Figure 2.

In the quantum repeater setup the optimal noise-free protocol is given by entanglement distillation between Alice and Charlie and Charlie and Bob, followed by entanglement swapping. This results in $\min\{E_D(\rho), E_D(\rho')\}$ for the quantum repeater rate, but in the key repeater setup the situation is less clear. In alternative to the

mentioned quantum repeater protocol, Alice and Charlie can distill private states instead, while Charlie and Bob distill maximally entangled states, and use the maximally entangled states to teleport Charlie's part of the private states to Bob. If $E_D(\rho')$ is larger than the private states size at Charlie's site, the rate of this “trivial” protocol equals $K_D(\rho)$ and thus it will be positive even for cases where ρ has zero distillable entanglement [18, 21]. In short, while for quantum repeaters the active area of research studies the effect of noisy operations, for quantum key repeaters there are open questions already with perfect operations.

We will consider a variation of the key repeater rate: the *one-way* key repeater rate R_D^\rightarrow also introduced in [26]. In this variation the classical communication is restricted. Alice and Bob can communicate normally, however their communication with the repeater station Charlie is restricted to be only one-way: only Charlie can send messages to Alice and Bob. In [26] the question was posed whether there exist nontrivial protocols beyond distillation and swapping, but only negative examples were found. Here we solidify this work by showing that for a large class of states and protocols, the one-way distillable entanglement is an upper bound on the one-way key repeater rate, and thus distillation and swapping are essentially optimal and far from the trivial upper bounds $K_D(\rho)$ and $K_D(\rho')$.

We first need a general upper bound on R_D^\rightarrow in terms of a regularized relative entropy from separable states restricted to measurements at Charlie's — Figure 3 — which follows from a bound given in [26].

Theorem 11. *For any pair of states ρ_{AC} and $\rho'_{C'B}$ and any separable state σ in $SEP_{A:CB}$ or $SEP_{AC:B}$ it holds*

$$R_D^\rightarrow(\rho, \rho') \leq D_C^\infty(\rho \otimes \rho' \| \sigma).$$

The proof is a straightforward modification of the proof of the bound in [26], therefore we leave it for the Supplementary Material (see Appendix Theorem 28).

As a direct application of Theorem 10 to Theorem 11, we now prove that for private states with separable key attacked state, there is almost no difference between the key repeater rate and the repeater rate. Since all private states are NPT (the partial transpose is non positive [20]), this will give first examples of NPT states with high distillable key but low one-way key repeater rate.

Corollary 12. *For any pair of key correlated states ρ and ρ' with at least one separable key attacked state, it holds:*

$$R_D^\rightarrow(\rho, \rho') \leq E_D^\rightarrow(\rho \otimes \rho').$$

Proof. Without loss of generality let $\hat{\rho}$ be separable. Apply Theorem 9 to Theorem 11 using $\hat{\rho} \otimes \hat{\rho}' \in A:CB$. \square

Note that inserting $\sigma = \hat{\rho} \otimes \rho'$ in Theorem 11 instead of $\hat{\rho} \otimes \hat{\rho}'$ (as done for Corollary 12) will often result in a tighter bound. E.g. in the case of $\rho = \rho'$ being maximally

entangled states, then the bound improves by a factor of two, going from $2E_D^{\rightarrow}(\rho)$ to $E_D^{\rightarrow}(\rho)$ and matching the lower bound. Nonetheless, the following example shows how the bound of Corollary 12 can be arbitrarily small even for states with perfect key.

Example 13. Consider the Bell private state of [18]:

$$\gamma = \frac{1}{2} \left(1 + \frac{1}{d}\right) \phi_+ \otimes \rho_{sym} + \frac{1}{2} \left(1 - \frac{1}{d}\right) \phi_- \otimes \rho_{asym}$$

where ρ_{sym} and ρ_{asym} are the symmetric and anti-symmetric states in $\mathbb{C}^d \otimes \mathbb{C}^d$ — the extreme Werner states [3] which are known to be data hiding states [14]. The distillable entanglement is upper bounded by the log-negativity [13]; for this state it gives the following immediate upper bound, which vanishes for large d :

$$R_D^{\rightarrow}(\gamma, \gamma) \leq 2E_D^{\rightarrow}(\gamma) \leq 2E_N(\gamma) = 2 \log \left(1 + \frac{1}{d}\right). \quad (15)$$

This Bell private state was implemented experimentally for $d = 2$ [22]. The key was distilled at a rate $K \approx 0.69$, enough to break the log-negativity upper bound at $E_N(\gamma) = \log \frac{3}{2} \approx 0.58$. However, because of the factor 2 in Equation (15), one would need an implementation with $d = 4$ at the same rate, in order to prove that the key repeater rate lies below it. Still, scaling up the implementation should be experimentally feasible, since in $d=4$ the gate used (swap) is tensor product of qubit gates. \blacktriangle

V. CONCLUSIONS

Corollary 12 bounds the key repeater rate of a restricted class of states and it shows that the key repeater rate can almost vanish for NPT states with high distillable key. In the Supplementary Material we show how this applies also to some PPT invariant states. Furthermore, Corollary 12 generalizes to all states for restricted protocols that first distill private states with separable key attacked state between the nodes and then try to repeat. In the Supplementary Material we define a new key repeater rate $R_D^{\rightarrow}(\rho, \rho')$ from these restricted protocols and we prove that for all ρ and ρ'

$$R_D^{\rightarrow}(\rho, \rho') \leq E_D^{\rightarrow}(\rho \otimes \rho'). \quad (16)$$

The restricted protocols still include one-way entanglement distillation and swapping, thus the new key repeater rate is still lower bounded by the minimum of

the one-way distillable entanglements. While being somewhat restrictive, to the best of our knowledge, all known one-way repeater protocols can be cast into this form.

We leave as open problem whether Equation (16) generalizes to all protocols including two way communication, namely whether

$$R_D(\rho, \rho') \leq E_D(\rho \otimes \rho')? \quad (17)$$

Such a result would show that all entangled states with zero distillable entanglement, including those with key, have zero key repeater rate. Another open problem, called the PPT² conjecture [27], asks whether swapping PPT states in all dimensions always yields separable states. If the conjecture is true, then it would imply that all PPT states have zero key repeater rate. In that, the results presented in this paper may be seen as support for the conjecture. Since our results are asymptotic in nature, they give a complementary view on the PPT² conjecture than the one offered by studying the swapping of specific pairs of states or in specific dimensions.

The connection made between key distillation, entanglement distillation and quantum data hiding raises the possibility of finding a rate at which data hiding states can be distilled H_D (that we refrain from defining formally). Namely, in performing entanglement distillation on private states, it may be possible to retain the undistillable correlations into data hiding states with zero distillable entanglement, so that

$$K_D(\rho) = H_D(\rho) + E_D(\rho)$$

and such that the data hiding states could be used for different purposes at a later point.

ACKNOWLEDGMENTS

We thank Alexander Müller-Hermes, Cécilia Lancien and Māris Ozols for helpful discussion. We acknowledge financial support from the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH), the European Research Council (ERC Grant Agreement no 337603), the Danish Council for Independent Research (Sapere Aude) and the Swiss National Science Foundation (project no PP00P2_150734).

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In this appendices we present a review of the background concepts used in this article, we show how to further apply our findings to more complex repeater scenarios and we provide more examples. We begin with a review of the generalized Bell states and of their properties, especially with respect to the bilateral CNOT which is fundamental to Construction 2. Then we analyze Bell private states further and present some minor properties.

After this, we move onto entanglement measures. First we review the various entanglement measures based on the relative entropy and its restriction to quantum measurements; this heavily relies on the work made in [24]. We then explain and give the explicit definition of all the distillation rates mentioned in this paper, including the various forms of distillable entanglement, distillable key and key repeater rate. We also discuss in details some known upper bounds on these distillation rates that were used in the main text.

With these concepts in place we present further applications of our results: the one-way key swapper (a novel repeater rate) and the single-copy repeater rate. Finally,

we give further examples of states with vanishing one-way key repeater rate, which include NPT states, PPT states and PPT invariant states.

Last but not least, the reader can find an exhaustive list of notations in Tables 7 and 8.

Appendix A: GENERALIZED BELL STATES

For two qubits ($\mathbb{C}^2 \otimes \mathbb{C}^2$) we can write the Bell states as bit flips and phase flips of the maximally entangled state. Namely, the maximally entangled state is

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The bit flip and phase flip gates are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we write the Bell states using powers of these matrices acting on one of the qubits. Choosing one qubit or the other yields different basis, we will choose to act always on the first one and so we have

$$|\phi_{ij}\rangle = (X^i Z^j \otimes \mathbb{1}) |\Phi\rangle \quad i, j = 0, 1$$

which gives the Bell states:

$$\begin{aligned} |\phi_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi\rangle \\ |\phi_{01}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\phi_{10}\rangle &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\ |\phi_{11}\rangle &= \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \end{aligned}$$

The generalized Bell states are defined in a similar way. Now we have two qudits ($\mathbb{C}^d \otimes \mathbb{C}^d$) whose maximally entangled state has been introduced previously:

$$|\Phi\rangle = \frac{1}{d} \sum_{i=0}^{d-1} |ii\rangle.$$

The unitary generalization of the bit and phase flip are:

$$X = \sum_{j=0}^{d-1} |j+1 \bmod d\rangle\langle j| \quad Z = \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j|$$

where $\omega = e^{i\frac{2\pi}{d}}$ is the d 'th root of unity. Just as before, the Bell states are now defined using powers of X and Z :

Definition 14 (Generalized Bell states).

$$|\phi_{ij}\rangle := (X^i Z^j \otimes \mathbb{1}) |\Phi\rangle$$

for $j, k = 0, \dots, d-1$.

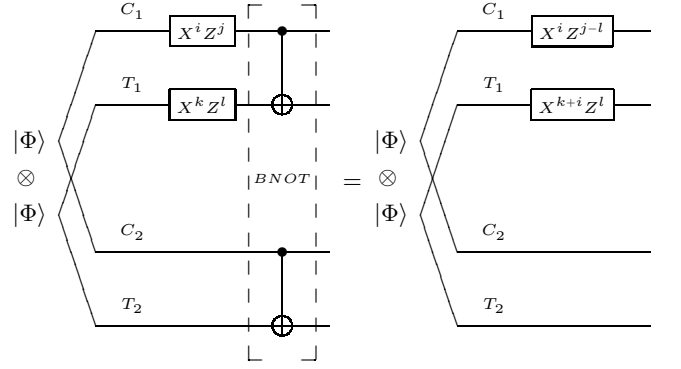


FIG. 4. Effect of the BNOT on Bell states — see Lemma 15.

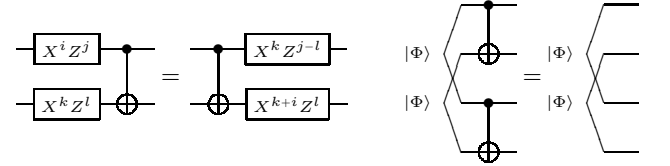


FIG. 5. Effect of commuting CNOT with arbitrary bit/phase flips. FIG. 6. Bell states with no bit/phase flips are invariant under BNOT.

We denote the density matrices of these states with:

$$\phi_{ij} = |\phi_{ij}\rangle\langle\phi_{ij}|.$$

Finally, it is now easy to show that the projector onto the maximally correlated subspace can be rewritten as a mixture of Bell states

$$\begin{aligned} \mathbb{1}_c &= \sum_i |ii\rangle\langle ii| = \sum_{ij} \delta_{ij} |ii\rangle\langle jj| \\ &= \sum_{ijk} \frac{1}{2^m} \omega^{k(i-j)} |ii\rangle\langle jj| = \sum_k \phi_{0k} \end{aligned}$$

and similarly for the maximally correlated state:

$$\tau_c = \frac{1}{2^m} \sum_i |ii\rangle\langle ii| = \frac{1}{2^m} \sum_j \phi_{0j}.$$

Bilateral CNOT (BNOT). The bilateral CNOT is a gate of four systems obtained by applying two CNOT gates, two of the systems will be the controls and the others will be the targets. In [5] it was called Bilateral XOR (BXOR) and it was defined only for qubit systems. Here we use the generalized CNOT on $\mathbb{C}^d \otimes \mathbb{C}^d$

$$CNOT = |a, a+b \bmod d\rangle\langle ab|$$

to obtain a straightforward generalization of the BXOR gate, the *bilateral CNOT*:

$$BNOT_{CT} = CNOT_{C_1 T_1} \otimes CNOT_{C_2 T_2}$$

where systems $C = C_1 C_2$ are the control qudits, systems $T = T_1 T_2$ are the target qudits and all qudits have the same size, i.e. $|C_1| = |C_2| = |T_1| = |T_2| = d$. Like in [5], our interest in the gate lies on its effect on Bell states. Notice how the $BNOT$ is an LO operation as long as the system is partitioned as $C_1 T_1 : C_2 T_2$.

Lemma 15.

$$BNOT \cdot |\phi_{ij}\rangle \otimes |\phi_{kl}\rangle = |\phi_{i,j-l}\rangle \otimes |\phi_{k+i,l}\rangle$$

Proof.

$$\begin{aligned} BNOT \cdot |\phi_{ij}\rangle \otimes |\phi_{kl}\rangle &= BNOT \cdot \frac{1}{2^m} \sum_{ab} \omega^{ja} |a+i, a\rangle \otimes \omega^{lb} |b+k, b\rangle \\ &= \frac{1}{2^m} \sum_{ab} \omega^{ja} |a+i, a\rangle \otimes \omega^{lb} |b+k+a+i, b+a\rangle. \end{aligned}$$

Now we make a change of variable $\tilde{b} = b + a$:

$$\begin{aligned} &= \frac{1}{2^m} \sum_{a\tilde{b}} \omega^{ja} |a+i, a\rangle \otimes \omega^{l(\tilde{b}-a)} |\tilde{b}+k+i, \tilde{b}\rangle \\ &= \frac{1}{2^m} \sum_{a\tilde{b}} \omega^{(j-l)a} |a+i, a\rangle \otimes \omega^{l\tilde{b}} |\tilde{b}+k+i, \tilde{b}\rangle \\ &= |\phi_{i,j-l}\rangle \otimes |\phi_{k+i,l}\rangle. \quad \square \end{aligned}$$

An alternative and maybe more intuitive way to prove Lemma 15, is to notice that $\Phi \otimes \Phi$ is invariant under the action of the $BNOT$ — see Figure 6 — and that $CNOT$ is a Clifford gate with a simple update rule — see Figure 5. Namely, it holds that

$$BNOT_{CT} \cdot (|\Phi\rangle_C \otimes |\Phi\rangle_T) = |\Phi\rangle_C \otimes |\Phi\rangle_T \quad (\text{A1})$$

and

$$CNOT \cdot X^i Z^j \otimes X^k Z^l = X^i Z^{j-l} \otimes X^{i+k} Z^l \cdot CNOT^\dagger. \quad (\text{A2})$$

Applying Equation (A1) and Equation (A2) to Definition 14 proves Lemma 15 as displayed in Figure 4. From Lemma 15, it follows in particular that

$$BNOT^\dagger \cdot \phi_{00} \otimes \phi_{0j} \cdot BNOT = \phi_{0j} \otimes \phi_{0j}$$

which is what we use in Lemma 4.

Appendix B: BELL PRIVATE STATES

Let us recall Definition 1 so that we can give some useful properties about them.

Definition 1. A Bell private state is any state

$$\gamma_{Bell}^m := \sum_{j=0}^{2^m-1} p_j \phi_{0j} \otimes \sigma_j$$

where σ_j are arbitrary orthogonal states.

Private state form. All Bell private states are private states. This can be proved either by checking that the measurement in $A_k B_k$ gives perfectly secure bits or by showing that they admit an expression as private states, here we show the latter.

Lemma 16. Any Bell private state is a private state with T and σ given by

$$T = \sum_{ij} |ij\rangle\langle ij| \otimes U_\sigma^i \quad \sigma = \sum_j p_j \sigma_j$$

and

$$U_\sigma = \sum_j \omega^j P_{\sigma_j} + P_{\sigma_\perp}$$

where P_{σ_j} are the projectors onto the supports of σ_j , P_{σ_\perp} is the remaining orthogonal projector, and U_σ^i is the i th power of U_σ .

P_{σ_\perp} plays no active role, it is needed only to complete T , so the shields do not need to span the whole space.

Proof. The following sequence of equalities proves that a state is a Bell private state, thus of the form $\sum p_j \phi_{0j} \otimes \sigma_j$, if and only if it is a private state, thus of the form $T(\Phi \otimes \sigma)T^\dagger$, with σ and T as above.

$$\begin{aligned} \gamma^m &= \sum_k p_k \phi_{0k} \otimes \sigma_k \\ &= \sum_{ijk} p_k \frac{1}{2^m} \omega^{ik} |ii\rangle\langle jj| \omega^{-jk} \otimes P_{\sigma_k} \sigma_k P_{\sigma_k} \\ &= \sum_{ijk\alpha\beta} \frac{1}{2^m} |ii\rangle\langle jj| \otimes (\omega^{ik} P_{\sigma_k}) \cdot (p_k \sigma_k) \cdot (\omega^{-jk} P_{\sigma_k}) \\ &= \sum_{ijk\alpha\beta} \frac{1}{2^m} |ii\rangle\langle jj| \otimes (\omega^{i\alpha} P_{\sigma_\alpha}) \cdot \\ &\quad \cdot \delta_{\alpha k} (p_k \sigma_k) \delta_{k\beta} \cdot (\omega^{-j\beta} P_{\sigma_\beta}) \\ &= \sum_{ijk\alpha\beta} \frac{1}{2^m} |ii\rangle\langle jj| \otimes (\omega^{i\alpha} P_{\sigma_\alpha}) \cdot (p_k \sigma_k) \cdot (\omega^{-j\beta} P_{\sigma_\beta}) \\ &= \sum_{ijk\alpha\beta} \frac{1}{2^m} |ii\rangle\langle jj| \otimes \\ &\quad \otimes (\omega^\alpha P_{\sigma_\alpha} + P_{\sigma_\perp})^i \sigma (\omega^\beta P_{\sigma_\beta} + P_{\sigma_\perp})^{-j} \\ &= \sum_{ij} \frac{1}{2^m} |ii\rangle\langle jj| \otimes U_\sigma^i \sigma U_\sigma^{-j} \\ &= \sum_{ij} (|i\rangle\langle i| \otimes U_\sigma^i) \cdot \left(\frac{1}{2^m} |ii\rangle\langle jj| \otimes \sigma \right) \cdot (|j\rangle\langle j| \otimes U_\sigma^{j\dagger}) \\ &= T (\Phi^m \otimes \sigma) T^\dagger \end{aligned}$$

where we used orthogonality of the shields in

$$P_{\sigma_\alpha} \sigma_k = \delta_{\alpha k} P_{\sigma_k} \sigma_k. \quad \square$$

Block form. For $|A_k| = |B_k| = 2$, any private state admits a *block form* [20], i.e. it can be written as:

$$\gamma^1 = \frac{1}{2} \begin{pmatrix} \sqrt{Y^\dagger Y} & 0 & 0 & Y^\dagger \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y & 0 & 0 & \sqrt{Y Y^\dagger} \end{pmatrix}$$

where Y is any opportune matrix of unit trace norm ($\|Y\|_1 = 1$). This can be easily seen by recalling that any matrix Y admits a singular value decomposition and the decomposition can be used to extract σ and the unitaries U_0 and U_1 in T . However, this does not work in higher dimension ($m = \log |A_k| > 1$) because then additional unitaries are needed to specify T .

This is not true for Bell private states, indeed, a Bell private state only needs to specify a single unitary U_σ . This allows one to write a block form for all private states by exploiting that U_σ and σ commute.

Corollary 17. γ_{Bell}^m is a Bell private state iff

$$\gamma_{Bell}^m = \frac{1}{2^m} \sum_{ij} |ii\rangle\langle jj| \otimes |Y| \left(\frac{Y}{|Y|} \right)^{i-j}$$

for some normal Y such that $\|Y\|_1 = 1$ and $Y^{2^m} \geq 0$.

Proof. Set $Y = \sigma U_\sigma$. Then $|Y| = \sigma$, $\frac{Y}{|Y|} = U_\sigma$ and

$$U_\sigma^i \sigma U_\sigma^{-j} = \sigma U_\sigma^{i-j} = |Y| \left(\frac{Y}{|Y|} \right)^{i-j}. \quad \square$$

In Corollary 17, Y^{-1} is intended as pseudo inversion of matrices so that Y need not to be full rank. Note how for $m = 1$ the corollary implies $Y = Y^\dagger$.

Key attacked state vs Marginals. Let us consider

$$\hat{\gamma} = \gamma_{A_k B_k} \otimes \gamma_{A_s B_s}$$

the product state of the key and shield marginals of a private state. For Bell private states this has a form similar to the key attacked state:

$$\hat{\gamma}_{Bell} = \sum_j p_j \phi_{0j} \otimes \sigma$$

which indeed gives $\hat{\gamma}_{Bell} = \hat{\gamma}_{Bell}$ for uniform $\{p_k\}$'s.

We can summarize the difference between $\hat{\gamma}$ and the key attacked state $\hat{\gamma}$ using quantum relative entropies ($D(\rho \parallel \sigma) = \text{Tr}[\rho \log \rho - \rho \log \sigma]$) as follows:

$$\begin{aligned} D(\gamma_{Bell}^m \parallel \hat{\gamma}_{Bell}^m) &= D(\Phi \parallel \tau_c) = m \\ D(\gamma_{Bell} \parallel \hat{\gamma}_{Bell}) &= I(A_k B_k : A_s B_s)_{\gamma_{Bell}} = H(\{p_k\}) \end{aligned}$$

where $I(A_k B_k : A_s B_s)$ is the quantum mutual information and $H(\{p_k\})$ is the entropy of $\{p_k\}$. Since γ_{Bell} commutes with both $\hat{\gamma}_{Bell}$ and $\hat{\gamma}_{Bell}$, these values are achieved also by performing a global measurement first and then computing the classical relative entropies [2, 30]. In both cases the optimal measurement operators are $\{M_{abj}\} = \{P_{\phi_{ab}} \otimes P_{\sigma_j}\}$.

Distillable Entanglement of $A_k B_k$. Another simplification involves the expression for the distillable entanglement in $A_k B_k$:

Lemma 18. For all Bell private states γ_{Bell}^m it holds

$$E_D(\gamma_{Bell, A_k B_k}^m) = m - H(\{p_k\})$$

where $\gamma_{Bell, A_k B_k}^m = \text{Tr}_{A_s B_s} \gamma_{Bell}^m$.

Proof. For all private states, the reduced state of $A_k B_k$ has support only on the maximally correlated subspace for which the distillable entanglement has been proven [23] to equal the hashing bound [17]:

$$E_D(\text{Tr}_{A_s B_s} \gamma) = H(B_k)_\gamma - H(A_k B_k)_\gamma.$$

For Bell private states we have:

$$\text{Tr}_{A_s B_s} \gamma_{Bell}^m = \sum_j p_j \phi_{0j}$$

thus the marginal of B_k will be completely mixed so the entropy of B_k will be maximal, while the entropy of $A_k B_k$ is the entropy of $\{p_k\}$:

$$E_D(\text{Tr}_{A_s B_s} \gamma_{Bell}^m) = m - H(\{p_k\}). \quad \square$$

Of particular interest is the case of uniform $\{p_k\}$, then:

$$E_D(\text{Tr}_{A_s B_s} \gamma_{Bell}) = 0$$

which is independent of m .

Appendix C: RELATIVE ENTROPIES

The *quantum relative entropy* between two states ρ and σ is defined as:

$$D(\rho \parallel \sigma) = \text{Tr} \rho [\log \rho - \log \sigma].$$

Some interesting entanglement measures are defined using this function. The *relative entropy of entanglement* is the relative entropy with respect to the set of separable states:

Definition 19 (Relative entropy of ent. [6]).

$$E_R(\rho) = \inf_{\sigma \in SEP} D(\rho \parallel \sigma)$$

This was generalized to the *relative entropy with respect to P* , where P is an arbitrary set of states:

Definition 20 (Relative entropy wrt P [24]).

$$E_R^P(\rho) = \inf_{\sigma \in P} D(\rho \parallel \sigma)$$

For an arbitrary set of measurements \mathbb{M} one defines the \mathbb{M} -*relative entropy* [24]. However, for now, we can allow any set of quantum maps \mathbb{L} (completely positive trace preserving maps), not just measurement, to make the definition more general:

Definition 21 (\mathbb{L} -relative entropy [24]).

$$D_{\mathbb{L}}(\rho \parallel \sigma) = \sup_{\Lambda \in \mathbb{L}} D(\mathcal{M}(\rho) \parallel \mathcal{M}(\sigma)) .$$

Combining optimization over quantum maps and states gives the \mathbb{L} -relative entropy with respect to P :

Definition 22 (\mathbb{L} -relative entropy wrt P [24]).

$$E_{R,\mathbb{L}}^P(\rho) = \inf_{\sigma \in P} D_{\mathbb{L}}(\rho \parallel \sigma) .$$

Regularization. Given a function on states f we consider the standard regularization

$$f^\infty(\rho) = \lim_{n \rightarrow \infty} f(\rho^{\otimes n})$$

whenever the limit is well defined. Similarly for functions g of two states we can consider a regularization

$$g^\infty(\rho, \sigma) = \lim_{n \rightarrow \infty} g(\rho^{\otimes n}, \sigma^{\otimes n})$$

again whenever the limit is well defined.

Fekete's lemma [1] guarantees that the regularization is well defined, at least for classes of partial measurements that we consider in Theorem 9 and Theorem 11. The lemma states that if a sequence D_n is superadditive, namely if it satisfies

$$D_{m+n} \geq D_n + D_m , \quad (\text{C1})$$

then $\frac{1}{n}D_n$ is convergent and $\lim_{n \rightarrow \infty} D_n = \sup_n \frac{D_n}{n}$.

This lemma guarantees that we can take the regularization of the various relative entropies defined above when P and \mathbb{M} satisfy some conditions. With this conditions we want to capture the properties of classes of maps like *LOCC* that are defined for states ρ and $\rho^{\otimes n}$ alike, even if they live in different spaces, and that are such that they map separable states to separable state, which are also defined irrespective of the underlying space of Alice and Bob. To make this rigorous we need to explicitly consider Hilbert spaces and the corresponding space of density matrices, let us denote them with \mathcal{H} and $\mathcal{D}(\mathcal{H})$ and let \mathcal{H} be finite dimensional.

We say that \mathbb{L} is a *class closed under tensor products* — or simply *class* — of quantum maps (or measurements) if it is a sequence of \mathbb{L}_n , each a set of quantum maps on $\mathcal{D}(\mathcal{H}^{\otimes n})$, such that:

$$\Lambda \in \mathbb{L}_m, \Gamma \in \mathbb{L}_n \Rightarrow \Lambda \otimes \Gamma \in \mathbb{L}_{m+n} \quad \forall m, n .$$

In such case it is easy to check that $D_{\mathbb{L}_{m+n}}(\rho^{\otimes m} \parallel \sigma^{\otimes n})$ satisfies Equation (C1):

$$\begin{aligned} D_{\mathbb{L}_{m+n}}(\rho^{\otimes m+n} \parallel \sigma^{\otimes m+n}) \\ \geq D_{\mathbb{L}_m}(\rho^{\otimes m} \parallel \sigma^{\otimes m}) + D_{\mathbb{L}_n}(\rho^{\otimes n} \parallel \sigma^{\otimes n}) \end{aligned}$$

and thus we can define:

Definition 23 (Regularized \mathbb{L} -relative entropy).

Let \mathbb{L} be a class of quantum maps closed under tensor products. Then , define:

$$D_{\mathbb{L}}^\infty(\rho \parallel \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} D_{\mathbb{L}_n}(\rho^{\otimes n} \parallel \sigma^{\otimes n}) .$$

We now restrict \mathbb{L} to be a class of \mathbb{M} measurements only, so that we can regularize $D_{\mathbb{M}}^P$. As done in [24], we also need to impose further conditions on P with respect to \mathbb{M} and on P itself. In what follows, let us denote with I a proper subset of $\{1 \dots n\}$.

We say that P is a *class closed under partial trace* — or simply *class* — of states, if it is a sequence of P_n , each a convex set of states in $\mathcal{D}(\mathcal{H}^{\otimes n})$, such that:

$$\sigma \in P_n \Rightarrow \text{Tr}_I \sigma \in P_{n-|I|} \quad \forall n, I$$

where Tr_I is the partial trace over the Hilbert spaces designated by I . Furthermore, let \mathbb{M} be a class of measurements, we say that P is *closed under \mathbb{M}* , if for all $n > m$, it holds that

$$\left\{ \begin{array}{l} \mathcal{M} \in \mathbb{M}_m \\ \sigma \in P_n \end{array} \right\} \Rightarrow \frac{\text{Tr}_I M_k^I \sigma}{\text{Tr} M_k^I \sigma} \in P_{n-m} \quad \forall k, |I| = m$$

where $\{M_k\}$ are the measurements operators of \mathcal{M} , and I designates on which system the measurement is acting so that $M_k^I \equiv M_k^I \otimes \mathbb{1}$, where the identity acts on the complement of I . These properties are enough to guarantee that the \mathcal{M} -relative entropy with respect to P also satisfies Equation (C1); we will restate here the original proof for the sake of completeness.

Lemma 24 ([24, Theorem 2(d)]). *Let P be a class of states and \mathbb{M} a class of measurements, such that P is closed under \mathbb{M} . Then:*

$$E_{R,\mathbb{M}_{m+n}}^P(\rho^{\otimes m+n}) \geq E_{R,\mathbb{M}_m}^P(\rho^{\otimes m}) + E_{R,\mathbb{M}_n}^P(\rho^{\otimes n}) .$$

Proof. For every ρ and every $\sigma \in P_{m+n}$, it holds

$$\begin{aligned} D_{\mathbb{M}_{m+n}}(\rho^{\otimes m+n} \parallel \sigma) \\ = \sup_{\mathcal{M} \in \mathbb{M}_{m+n}} D(\mathcal{M}(\rho^{\otimes m+n}) \parallel \mathcal{M}(\sigma)) \\ \geq \sup_{\substack{\mathcal{M} \in \mathbb{M}_m \\ \mathcal{N} \in \mathbb{M}_n}} D(\mathcal{M}(\rho^{\otimes m}) \otimes \mathcal{N}(\rho^{\otimes n}) \parallel \mathcal{M} \otimes \mathcal{N}(\sigma)) \end{aligned}$$

Now, we make the measurement outcome and probabilities of \mathcal{M} explicit. Let I be the systems on which \mathcal{M} is acting. Then:

$$\begin{aligned} D_{\mathbb{M}_{m+n}}(\rho^{\otimes m+n} \parallel \sigma) \\ \geq \sup_{\substack{\mathcal{M} \in \mathbb{M}_m \\ \mathcal{N} \in \mathbb{M}_n}} D(\mathcal{M}(\rho^{\otimes m}) \otimes \mathcal{N}(\rho^{\otimes n}) \parallel \sum_k |k\rangle\langle k| \otimes \mathcal{N}(\text{Tr}_I M_k \sigma)) \\ = \sup_{\substack{\mathcal{M} \in \mathbb{M}_m \\ \mathcal{N} \in \mathbb{M}_n}} D(\mathcal{M}(\rho^{\otimes m}) \otimes \mathcal{N}(\rho^{\otimes n}) \parallel \sum_k \text{Tr}[M_k \sigma] |k\rangle\langle k| \otimes \mathcal{N}\left(\frac{\text{Tr}_I[M_k \sigma]}{\text{Tr}[M_k \sigma]}\right)) \end{aligned}$$

Let now σ_I be the reduced state of σ on systems I , define $\sigma_k^{\mathcal{M}} = \frac{\text{Tr}_I[M_k \sigma]}{\text{Tr}[M_k \sigma]}$ and recall that $\sigma_k^{\mathcal{M}} \in P_n$ by assumption. Then, we can rewrite the above as:

$$\begin{aligned}
D_{\mathbb{M}_{m+n}}(\rho^{\otimes m+n} \| \sigma) &\geq \sup_{\mathcal{M} \in \mathbb{M}_m} D(\mathcal{M}(\rho^{\otimes m}) \| \mathcal{M}(\sigma_I)) \\
&\quad + \sup_{\substack{\mathcal{M} \in \mathbb{M}_m \\ \mathcal{N} \in \mathbb{M}_n}} \sum_k \text{Tr}[M_k \rho^{\otimes m}] D(\mathcal{N}(\rho^{\otimes n}) \| \mathcal{N}(\sigma_k^{\mathcal{M}})) \\
&\geq \sup_{\mathcal{M} \in \mathbb{M}_m} D(\mathcal{M}(\rho^{\otimes m}) \| \mathcal{M}(\sigma_I)) \\
&\quad + \sup_{\substack{\mathcal{M} \in \mathbb{M}_m \\ \mathcal{N} \in \mathbb{M}_n}} D(\mathcal{N}(\rho^{\otimes n}) \| \sum_k \text{Tr}[M_k \rho^{\otimes m}] \mathcal{N}(\sigma_k^{\mathcal{M}}))
\end{aligned}$$

where the last step follow by the joint convexity of the relative entropy. Now set $\tilde{\sigma} = \sum_k \text{Tr}[M_k \rho^{\otimes m}] \mathcal{N}(\sigma_k^{\mathcal{M}})$ and notice that by convexity assumption $\tilde{\sigma} \in P_n$. Therefore:

$$\begin{aligned}
D_{\mathbb{M}_{m+n}}(\rho^{\otimes m+n} \| \sigma) &\geq \sup_{\mathcal{M} \in \mathbb{M}_m} D(\mathcal{M}(\rho^{\otimes m}) \| \mathcal{M}(\sigma_I)) \\
&\quad + \sup_{\mathcal{N} \in \mathbb{M}_n} D(\mathcal{N}(\rho^{\otimes n}) \| \mathcal{N}(\tilde{\sigma})) \\
&= D_{\mathbb{M}_m}(\rho^{\otimes m} \| \sigma_I) + D_{\mathbb{M}_n}(\rho^{\otimes n} \| \tilde{\sigma}) \\
&\geq \inf_{\sigma \in P_m} D_{\mathbb{M}_m}(\rho^{\otimes m} \| \sigma) + \inf_{\sigma \in P_n} D_{\mathbb{M}_n}(\rho^{\otimes n} \| \sigma) \\
&= E_{R, \mathbb{M}_m}^P(\rho^{\otimes m}) + E_{R, \mathbb{M}_n}^P(\rho^{\otimes n}) \quad \square
\end{aligned}$$

Thanks to this lemma the following regularization is now well defined.

Definition 25 (Regularized \mathbb{M} -relative entropy wrt P). Let P be a class of states and let \mathbb{M} be a class of measurements such that P is closed under \mathbb{M} . Define:

$$E_{R, \mathbb{M}}^{\infty, P}(\rho) = \lim_{n \rightarrow \infty} \frac{1}{n} E_{R, \mathbb{M}}^P(\rho^{\otimes n}).$$

Notice that the class of PPT states is closed under the class of PPT measurements and consequently under all subclasses of measurements, like separable and LOCC measurements. Similarly the class of separable states is closed under the class of separable measurements, and consequently under LOCC measurements and its variations. Thus Definition 25 is always well defined for the above combinations of states and measurements.

As it is usually done, we will omit the fact that the classes of states and quantum maps/measurements are actually sequences of such regularized quantities, therefore we will drop the index n . We will use σ_P and $\mathcal{M}_{\mathbb{M}}$ as short hand notation for $\sigma \in P$ and $\mathcal{M} \in \mathbb{M}$, respectively. See Table 7 for a more detailed list of symbols and notations.

Finally, we highlight that we do not have yet a general Definition 25 for arbitrary classes of quantum maps. This in particular includes classes of partial measurements, where only some party are force to measure their systems. Already for partial measurements, the only regularized definition that we can use for now is Definition 23.

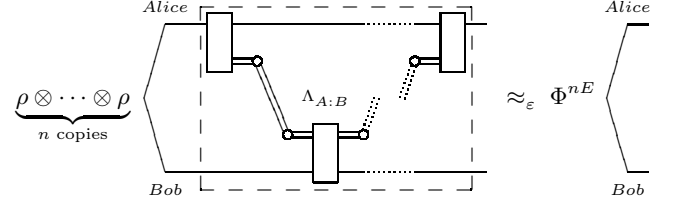


FIG. 9. Quantum circuit for entanglement distillation. The circuit in the dotted box the map $\Lambda \in \text{LOCC}_{A:B}$, optimized to output Φ^{nE} with the highest nE possible.

Appendix D: ENTANGLEMENT AND KEY DISTILLATION

When performing distillation, the goal is to approximate a desired output state by acting on the input via the allowed operations. Usually, trace norm distance is used to quantify the approximation between two states. We will use the expression

$$\rho \approx_{\epsilon} \sigma$$

to denote

$$\|\rho - \sigma\|_1 \leq \epsilon$$

between arbitrary states ρ and σ .

Then, formally, the distillable entanglement of a state ρ is defined as the rate at which maximally entangled states can be distilled under bipartite LOCC [13]:

$$E_D(\rho_{AB}) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \{E : \Lambda(\rho_{AB}^{\otimes n}) \approx_{\epsilon} \Phi_{AB}^{nE}\}$$

as illustrated in Figure 9.

The distillable key is defined as the rate at which perfect secret bits can be distilled. However it also equals the rate at which private states can be distilled under bipartite LOCC [18]:

$$K_D(\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \{K : \Lambda(\rho^{\otimes n}) \approx_{\epsilon} \gamma^{nK}\}.$$

Note how the expression is almost the same as for distillable entanglement, the only difference are the desired output states. Because of the reversible protocol of Construction 2 it is now also possible to write the distillable key as the rate at which Bell private states can be distilled:

$$K_D(\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \{K : \Lambda(\rho^{\otimes n}) \approx_{\epsilon} \gamma_{\text{Bell}}^{nK}\}.$$

This are only some of the rates that can be defined. While the target state defines the kind of resource being measured (pure entanglement, key, ...), changing the available protocols produces variations of these quantities that reflect different scenarios, like in the case of the one-way distillable entanglement. This is:

$$E_D^{\rightarrow}(\rho) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup \{E : \Lambda(\rho^{\otimes n}) \approx_{\epsilon} \Phi^{nE}\}$$

Objects	Shorthands	Meanings
Alice, Bob	A, B	The two parties.
$A_k B_k$		The key systems of Alice and Bob, $ A_k = B_k $.
m		The size of the key systems in bits, $m = \log A_k $ (with $\log \equiv \log_2$).
$A_s B_s$		The shield systems of Alice and Bob. In most the examples we use $ A_s = B_s = d$, but this is not a requirement.
$\rho, \hat{\rho}$		A state and its key attacked state (when the state has key systems).
\mathcal{M}		A measurement.
Λ		A map/protocol.
LO_A, LO_B		Local operations, i.e. any quantum channel acting only on Alice's and Bob's systems respectively.
$LOCC_{A:B}, LOCC_{A \rightarrow B}$		Local Operations with two-way and one-way Classical Communication.
$\mathcal{M} \in LO_A$	\mathcal{M}_A	A measurement at Alice's.
$\mathcal{M} \in LOCC_{A:B}$ $\mathcal{M} \in LOCC_{A \rightarrow B}$	$\mathcal{M}_{A:B}$ $\mathcal{M}_{A \rightarrow B}$	A measurement in $LOCC_{A:B}$ and $LOCC_{A \rightarrow B}^{\rightarrow}$ respectively.
$\mathcal{M}_A \in LOCC_{A:B}$		A partial measurement in $LOCC_{A:B}$, Alice needs to measure, but not Bob.
$\Lambda \in LOCC_{A:B}$ $\Lambda \in LOCC_{A \rightarrow B}$	$\Lambda_{A:B}$ $\Lambda_{A \rightarrow B}$	A protocol in $LOCC_{A:B}$ and $LOCC_{A \rightarrow B}^{\rightarrow}$ respectively
D_A		Relative entropy restricted to partial measurements $\mathbb{L} = \mathbb{M} \otimes \text{id}_A$, where $\mathbb{M} = LO_A$.
$D_{A \rightarrow B}$		Relative entropy restricted to measurements in $\mathbb{M} = LOCC_{A \rightarrow B}$.
$E_{R,A \rightarrow B}$		Relative entropy with measurements in $\mathbb{M} = LOCC_{A \rightarrow B}$ and $P = SEP_{A:B}$.

TABLE 7. General notation for states, maps, measurements, parties, etc.

Objects	Shorthands	Meanings
C, C'	\mathbf{C}	The parties played by Charlie. C and C' share entanglement with Alice and Bob respectively.
C_k, C'_k	\mathbf{C}_k	The key systems of Charlie.
C_s, C'_s	\mathbf{C}_s	The shield systems of Charlie.
$\sigma \in SEP_{A:C:C':B}$	$\sigma_{A:C:C':B}$	Quadri-separable states of Alice, Bob, and Charlie's parties.
$\sigma \in SEP_{A:CB}$	$\sigma_{A:CB}$	Separable states between Alice and the joint systems of Charlie and Bob.
$\sigma \in SEP_{AC:B}$	$\sigma_{AC:B}$	Separable states between Bob and the joint systems of Charlie and Alice.
$LOCC_{A:C:B}$		Tripartite LOCC of Alice, Charlie and Bob.
$LOCC_{C \rightarrow AB}$		Tripartite LOCC with only one-way communication from Charlie to Alice/Bob. Alice and Bob can communicate freely with each other.
$\mathcal{M} \in LO_C$	\mathcal{M}_C	A partial measurement at Charlie's.
$\mathcal{M} \in LOCC_{A:C:B}$ $\mathcal{M} \in LOCC_{C \rightarrow AB}$	$\mathcal{M}_{A:C:B}$ $\mathcal{M}_{C \rightarrow AB}$	A measurement in $LOCC_{A:C:B}$ and $LOCC_{C \rightarrow AB}$ respectively.
$\Lambda \in LOCC_{A:C:B}$ $\Lambda \in LOCC_{C \rightarrow AB}$	$\Lambda_{A:C:B}$ $\Lambda_{C \rightarrow AB}$	A protocol in $LOCC_{A:C:B}$ and $LOCC_{C \rightarrow AB}$ respectively.
D_C		Relative entropy restricted to partial measurements $\mathbb{L} = \mathbb{M} \otimes \text{id}_{AB}$, where $\mathbb{M} = LO_C$.
$D_{C \rightarrow AB}$		Relative entropy restricted to measurements in $\mathbb{M} = LOCC_{C \rightarrow AB}$.
$E_{R,C \rightarrow AB}^{A:C:C':B}$		Relative entropy with $P = SEP_{A:C:C':B}$ and measurements in $\mathbb{M} = LOCC_{C \rightarrow AB}$.
$E_{R,C \rightarrow AB}^{A:CB}$ $E_{R,C \rightarrow AB}^{AC:B}$		Relative entropy with $P = SEP_{A:CB}, SEP_{AC:B}$ and measurements in $\mathbb{M} = LOCC_{C \rightarrow AB}$.
$E_{R,MC \rightarrow AB}^{A:CB}$ $E_{R,MC \rightarrow AB}^{AC:B}$		Relative entropy with $P = SEP_{A:CB}, SEP_{AC:B}$ and $\mathbb{L} = LOCC_{MC \rightarrow AB}$, which are LOCC protocols of $C:AB$ followed by a measurement at Charlie.

TABLE 8. Additional notation regarding the repeater setups.

where the maps are restricted to one-way *LOCC*. Just like there is a regularized *LOCC* restricted relative entropy of entanglement upper bound on E_D , the same proof carry over to E_D^\rightarrow by simply restricting to one-way *LOCC*. Here below, we give an explicit proof of such bound which mirrors the one for E_D of [25].

Lemma 26 ([25]). *For any state ρ it holds:*

$$D_A^\infty(\rho \parallel \sigma) \geq E_{R,A \rightarrow B}^\infty(\rho) \geq E_D^\rightarrow(\rho)$$

for all separable states σ .

It should be understood that the direction of the communication in the measurement must be the same as the direction in the distillation protocol.

Proof. The proof of $E_{R,A \rightarrow B}^\infty(\rho) \geq E_D^\rightarrow(\rho)$ is a direct mirror the same result for *LOCC* operations (two way) from [25]. We need two results from the same article, the value of $E_{R,A \rightarrow B}$ on maximally entangled states [25, Proposition 4]:

$$\begin{aligned} E_{R,A \rightarrow B}(\Phi^m) &= \log(2^m + 1) - 1 \\ &= m + \log(1 + 2^{-m}) - 1, \end{aligned}$$

and its asymptotic continuity [25, Proposition 3]:

$$|E_{R,A \rightarrow B}(\varrho) - E_{R,A \rightarrow B}(\varsigma)| \leq 2\varepsilon \log \frac{3|AB|}{\varepsilon}$$

for states satisfying $\|\varrho - \varsigma\|_1 = \varepsilon \leq e^{-1}$. Now, for all ε and for all n let Λ be the optimal one-way distillation map of $E_D^\rightarrow(\rho)$. Then:

$$\begin{aligned} &\frac{1}{n} E_{R,A \rightarrow B}(\rho^{\otimes n}) \\ &\geq \frac{1}{n} E_{R,A \rightarrow B}(\Lambda(\rho^{\otimes n})) \\ &\geq \frac{1}{n} E_{R,A \rightarrow B}(\Phi^{nE}) - \frac{1}{n} 2\varepsilon \log \frac{6 \cdot 2^{2nE}}{\varepsilon} \\ &= E + \frac{1}{n} \log \left(1 + \frac{1}{2^{nE}} \right) - \frac{1}{n} - 4\varepsilon E - \frac{2\varepsilon}{n} \log \frac{6}{\varepsilon} \\ &\xrightarrow{n \rightarrow \infty} E - 4\varepsilon E. \end{aligned}$$

Taking the limit $\varepsilon \rightarrow 0$ ends the first part of the proof.

We now prove $D_A^\infty(\rho \parallel \sigma) \geq D_{A \rightarrow B}^\infty(\rho \parallel \sigma)$. This follows because the optimization in D_A is made over a larger class of maps, because Bob is not necessarily measured. More precisely, any measurement in $\mathcal{M} \in \text{LOCC}_{A \rightarrow B}$, can always be written as a measurement $\mathcal{M}' \in \text{LO}_A$ followed by a measurement on Bob conditioned on Alice's outcome [17], namely a global measurement \mathcal{M}'' acting on Bob and Alice's measurement outcome. The communication is implicit in the fact that \mathcal{M}'' has received the outcome of \mathcal{M}' and is treating $\mathcal{M}'(\rho)$ as a global state:

$$\mathcal{M}(\rho) = \mathcal{M}'' \circ (\text{id} \otimes \mathcal{M}')(\rho)$$

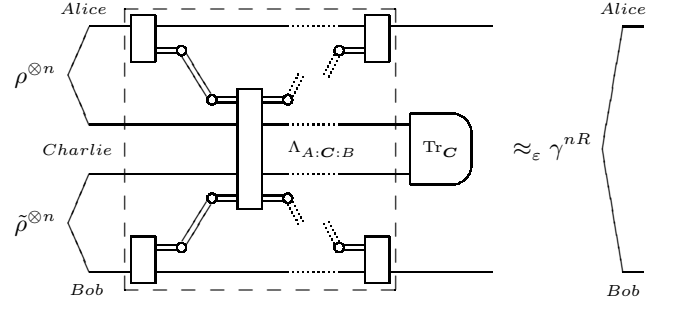


FIG. 10. Distillation of key in a single node repeater.

By the monotonicity of the relative entropy we thus have:

$$\begin{aligned} D(\mathcal{M}(\rho) \parallel \mathcal{M}(\sigma)) &= D(\mathcal{M}'' \circ \mathcal{M}'(\rho) \parallel \mathcal{M}'' \circ \mathcal{M}'(\sigma)) \\ &\leq D(\mathcal{M}'(\rho) \parallel \mathcal{M}'(\sigma)). \end{aligned}$$

Thus:

$$D_A(\rho \parallel \sigma) \geq D_{A \rightarrow B}(\rho \parallel \sigma). \quad \square$$

Appendix E: KEY REPEATER

The key repeater rate from [26] is also defined as a rate at which private states can be distilled, similar to the rates introduced above. However, we need to switch from bipartite *LOCC* maps to tripartite *LOCC* maps, we need to trace Charlie ($C = CC'$) at the end of the protocol and we have two inputs in tensor product instead of a single one — see also Figure 10. Formally,

$$R_D(\rho, \rho') = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \{ R : \text{Tr}_C \Lambda((\rho \otimes \rho')^{\otimes n}) \approx_\varepsilon \gamma^{nR} \},$$

which again equals

$$R_D(\rho, \rho') = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \{ R : \text{Tr}_C \Lambda((\rho \otimes \rho')^{\otimes n}) \approx_\varepsilon \gamma_{\text{Bell}}^{nR} \}.$$

Let us now indicate with $\text{LOCC}_{C \rightarrow A:B}$ the tripartite *LOCC* protocols that have the communication between Charlie and Alice/Bob restricted to be one-way from Charlie. The corresponding one-way key repeater rate is then defined as:

$$R_D^\rightarrow(\rho, \rho') = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \{ R : \text{Tr}_C \Lambda((\rho \otimes \rho')^{\otimes n}) \approx_\varepsilon \gamma^{nR} \}.$$

Lemma 27.

$$R_D^\rightarrow(\rho, \rho') = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \{ R : \Lambda \circ \mathcal{M}((\rho \otimes \rho')^{\otimes n}) \approx_\varepsilon \gamma^{nR} \}.$$

Proof. An arbitrary one-way protocol $\Lambda' \in \text{LOCC}_{C \rightarrow A:B}$ consists of an instrument on C followed by an $\text{LOCC}_{A:B}$ protocol that is allowed to

act on the classical part of the instrument outcome (the communication). Let I be the instrument from \mathcal{C} to \mathcal{CM} , where M is now the classical register. Let $\Lambda \in LOCC_{M:A:B}$ be the second part of the protocol. Then:

$$\begin{aligned} \text{Tr}_{\mathcal{C}} \Lambda' &= \text{Tr}_{\mathcal{C}} \circ (\text{id}_{\mathcal{C}} \otimes \Lambda) \circ (I \otimes \text{id}_{AB}) \\ &= \Lambda \circ (\text{Tr}_{\mathcal{C}} I \otimes \text{id}_{AB}) \\ &= \Lambda \circ (\mathcal{M} \otimes \text{id}_{AB}) \end{aligned}$$

where we used that tracing the quantum part of the instrument gives a measurement $\mathcal{M} = \text{Tr}_{\mathcal{C}} I$. Since M is classical we have $LOCC_{M:A:B} \equiv LOCC_{MA:B}$. Because every pair $(\mathcal{M}_{\mathcal{C}}, \Lambda_{A:B})$ also defines a map in $LOCC_{\mathcal{C} \rightarrow A:B}$, we have equality in the claim. \square

We will now provide the proof that we omitted from the main text after recalling the related statement.

Theorem 28 (Main text Theorem 11). *For any pair of states ρ and ρ' and any separable state σ in $SEP_{A:CB}$ or $SEP_{AC:B}$ it holds*

$$R_D^{\rightarrow}(\rho, \rho') \leq D_{\mathcal{C}}^{\infty}(\rho \otimes \rho' \parallel \sigma).$$

Proof. This is a consequence of a Theorem in [26] that states that for any such ρ , ρ' and σ it holds

$$R_D^{\rightarrow}(\rho, \rho') \leq E_{R, \mathcal{C} \rightarrow AB}^{\infty, A:C':B}(\rho \otimes \rho')$$

The direct way of proving the claim is to first adapt the proof of [26] to obtain:

$$\begin{aligned} R_D^{\rightarrow}(\rho, \rho') &\leq E_{R, \mathcal{C} \rightarrow AB}^{\infty, A:CB}(\rho \otimes \rho') \\ R_D^{\rightarrow}(\rho, \rho') &\leq E_{R, \mathcal{C} \rightarrow AB}^{\infty, AC:B}(\rho \otimes \rho'), \end{aligned}$$

then restrict the optimization over tensor product separable states $\sigma^{\otimes n}$ and finally remove the measurement on the receiver side AB as done in Lemma 26.

However, we think it is instructive to see a direct proof without measurements on AB but with the use of Lemma 27. Without loss of generality let σ in $SEP_{A:CB}$. Let \mathcal{M} and Λ be the one-way distillation protocols for given ε and n as in Lemma 27. Then, for any σ :

$$\begin{aligned} \frac{1}{n} D_{\mathcal{C}}((\rho \otimes \rho')^{\otimes n} \parallel \sigma^{\otimes n}) \\ &\geq \frac{1}{n} D(\mathcal{M}((\rho \otimes \rho')^{\otimes n}) \parallel \mathcal{M}(\sigma^{\otimes n})) \\ &\geq \frac{1}{n} D(\Lambda \circ \mathcal{M}((\rho \otimes \rho')^{\otimes n}) \parallel \Lambda \circ \mathcal{M}(\sigma^{\otimes n})) \end{aligned}$$

by monotonicity of the relative entropy. Notice that $\Lambda \circ \mathcal{M}(\sigma^{\otimes n}) \in SEP_{A:B}$ if $\sigma \in SEP_{A:CB}$. At this point the state will be ε -close to a private state γ^{nR} which, by definition, is a twisted version of a maximally entangled state Φ^{nR} . Let us denote by \mathcal{T} the map that inverts the twisting unitary and traces the shield. Then, by monotonicity of the trace distance, we have:

$$\mathcal{T} \circ \Lambda \circ \mathcal{M}((\rho \otimes \rho')^{\otimes n}) \approx_{\varepsilon} \Phi^{nR}.$$

Furthermore, while $\Lambda \circ \mathcal{M}(\sigma^{\otimes n})$ might not be separable anymore, \mathcal{T} will still map $SEP_{A:B}$ into a convex set $\mathcal{T}(SEP_{A:B})$. Let us denote by $\tilde{\sigma}_{\mathcal{T}(A:B)}$ a state in this set. Again by monotonicity of the relative entropy, we can then write:

$$\begin{aligned} \frac{1}{n} D_{\mathcal{C}}((\rho \otimes \rho')^{\otimes n} \parallel \sigma^{\otimes n}) \\ &\geq \frac{1}{n} D(\mathcal{T} \circ \Lambda \circ \mathcal{M}((\rho \otimes \rho')^{\otimes n}) \parallel \mathcal{T} \circ \Lambda \circ \mathcal{M}(\sigma^{\otimes n})) \\ &\geq \frac{1}{n} \inf_{\tilde{\sigma}_{\mathcal{T}(A:B)}} D(\mathcal{T} \circ \Lambda \circ \mathcal{M}((\rho \otimes \rho')^{\otimes n}) \parallel \tilde{\sigma}) \\ &\geq \frac{1}{n} \inf_{\tilde{\sigma}_{\mathcal{T}(A:B)}} D(\Phi^{nR} \parallel \tilde{\sigma}) - \frac{1}{n} 2\varepsilon \log \frac{2^{2nR+2}}{\varepsilon} \\ &\geq R - 4\varepsilon R - \frac{1}{n} 2\varepsilon \log \frac{4}{\varepsilon} \\ &\xrightarrow{n \rightarrow \infty} R - 4\varepsilon R \end{aligned}$$

where we used the asymptotic continuity of the relative entropy [9], and Lemma 7 from [20]. Taking the limit $\varepsilon \rightarrow 0$ and the infimum over σ concludes the proof. \square

The original version uses states σ separable in the quadri-partite cut $A:C':B$. However, it is immediate to see why the argument works also for $A:CB$ and $AC:B$: the only thing required in the proof is the separability of $\Lambda'(\sigma)$ in $A:B$ for any distillation protocol Λ' .

Appendix F: KEY SWAPPER

Before we begin to talk about key swapping protocols we need to introduce some definitions.

The class of Bell private states is not the only restriction one can make to the class of private states. The class of *irreducible* private states was defined in [20]:

Definition 29 (Irreducible private states [20]).

A private state γ^m is called irreducible if $K_D(\gamma^m) = m$.

These are the private states that are actually interesting in the definition of distillable key because they are the outcomes of the optimal distillation protocols. However, the only feasible way to prove that a private state is irreducible is to upper bound the distillable key via some other entanglement measure. For example one can use the relative entropy of entanglement, which in [20] it was shown to give a further upper bound in terms of

the relative entropy of entanglement of the key attacked state:

$$K_D(\gamma^m) \leq E_R(\gamma) \leq m + E_R(\hat{\gamma}). \quad (\text{F1})$$

In light of this technique and considering that we need to require that $\hat{\gamma}$ is separable to argue that $E_D = E_{R,LOCC}^\infty$, it is sensible to introduce the following definition:

Definition 30 (Strictly irreducible private states). We say a private state γ is strictly irreducible if $\hat{\gamma}$ is separable. We will denote these states with $\langle \gamma \rangle$ or γ^{sm} .

Of course, all strictly irreducible private states are irreducible. Indeed, these are all the private states for which we can prove $K_D(\gamma^{sm}) = m$ via Equation (F1). A simple example are all private states for which σ is the maximally mixed state τ ; indeed, we find immediately that

$$\begin{aligned} \hat{\gamma}^m &= \frac{1}{2^m} \sum_i |ii\rangle\langle ii| \otimes U_i \tau U_i^\dagger \\ &= \frac{1}{2^m} \sum_i |ii\rangle\langle ii| \otimes \tau U_i U_i^\dagger \\ &= \frac{1}{2^m} \sum_i |ii\rangle\langle ii| \otimes \tau \\ &= \tau_c \otimes \tau \end{aligned}$$

is always separable.

We now show that the one-way distillable entanglement upper bounds the one-way key repeater rate of all protocols that first distill strictly irreducible private states with Charlie and then try to apply a general repeater protocol. First, we define a rate for such protocols.

Definition 31 (One-way key swapping rate).

For all bipartite ρ and $\tilde{\rho}$, we define the one-way key swapping rate achieved with one-way key swapping protocols as:

$$\mathcal{R}_D^\rightarrow(\rho, \tilde{\rho}) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\substack{\Gamma_{C \rightarrow A} \\ \Gamma_{C' \rightarrow B} \\ \tilde{\Gamma}_{C \rightarrow A:B}}} \left\{ R : \begin{array}{l} \text{Tr}_C \Lambda(\gamma^{nr} \otimes \gamma^{m\tilde{r}}) \approx_\delta \gamma^{nR} \\ \Gamma(\rho^{\otimes n}) \approx_\varepsilon \gamma^{nr} \\ \tilde{\Gamma}(\tilde{\rho}^{\otimes n}) \approx_\varepsilon \gamma^{m\tilde{r}} \end{array} \right\}.$$

Then we can state the result:

Theorem 32.

$$\mathcal{R}_D^\rightarrow(\rho, \tilde{\rho}) \leq E_D^\rightarrow(\rho \otimes \tilde{\rho})$$

Proof. First notice that the tensor product of two strictly irreducible private states is still a strictly irreducible private state, i.e. $\gamma^{a\omega} \otimes \gamma^{b\omega} = \gamma^{(a+b)\omega}$.

Then, for the sake of the proof, let us introduce the following convenient bold shorthand notation:

$$\begin{array}{ll} \mathbf{r} = r + \tilde{r} & \mathbf{\rho} = \rho \otimes \tilde{\rho} \\ \gamma^{nr} = \gamma^{nr} \otimes \gamma^{m\tilde{r}} & \mathbf{\Gamma} = \Gamma \otimes \tilde{\Gamma} \\ \varepsilon = \varepsilon + \tilde{\varepsilon} & \lim := \lim_{\varepsilon \rightarrow 0} \lim_{\varepsilon, \tilde{\varepsilon} \rightarrow 0} \end{array}$$

then

$$\begin{aligned} &\|\mathbf{\Gamma}(\rho^{\otimes n}) - \gamma^{nr}\|_1 \\ &= \|\mathbf{\Gamma}(\rho^{\otimes n}) \otimes \tilde{\Gamma}(\tilde{\rho}^{\otimes n}) - \gamma^{nr} \otimes \gamma^{m\tilde{r}}\|_1 \\ &\leq \varepsilon + \tilde{\varepsilon} = \varepsilon. \end{aligned} \quad (\text{F2})$$

We also define:

$$\mathcal{E} = \mathcal{E}_{A_k C_k} \otimes \text{id}_{A_s C_s} \otimes \mathcal{E}_{B_k C'_k} \otimes \text{id}_{B_s C'_s}$$

where \mathcal{E} is the reversible map from Lemma 4.

Just like in the proof of Lemma 5, we exploit the idea of using the distillable entanglement as an upper bound on the protocol that performs a measurement on the shield followed by hashing. We just need some steps to adapt it to the key swapping rate definition. First, since the one-way distillable entanglement is a one-way LOCC monotone and $\mathcal{E} \circ \mathbf{\Gamma}$ is one-way LOCC:

$$\begin{aligned} E_D^\rightarrow(\rho) &= \lim_{n \rightarrow \infty} \frac{1}{n} E_D^\rightarrow(\rho^{\otimes n}) \\ &\geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\mathbf{\Gamma}} \frac{1}{n} E_D^\rightarrow(\mathcal{E} \circ \mathbf{\Gamma}(\rho^{\otimes n})) \end{aligned}$$

We consider the protocol that performs first a measurement \mathcal{M} on Charlie's shield systems and after distills by hashing. Then, we can further lower bound with:

$$E_D^\rightarrow(\rho) \geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\mathbf{\Gamma}} \frac{1}{n} \sup_{\mathcal{M}} I(\mathbf{C}_k)_{ABM} \mathcal{M} \circ \mathcal{E} \circ \mathbf{\Gamma}(\rho^{\otimes n})$$

where $I(X)Y = H(Y) - H(XY)$ is the coherent information, $\mathbf{C}_k = C_k C'_k$ and $AB = A_k B_k A_s B_s$. Now we want to change the approximate private states into exact private states using asymptotic continuity of the conditional entropy [28]. In the form of [29], it says that for arbitrary bipartite states such that $\varepsilon \geq \|\rho_{XY} - \sigma_{XY}\|_1$ it holds:

$$|I(X)Y_\rho - I(X)Y_\sigma| \leq \varepsilon \log |X| + \eta(1 + \frac{\varepsilon}{2}) - \eta(\frac{\varepsilon}{2})$$

where $\eta(x) = x \log x$.

Since from Equation (F2) we have

$$\varepsilon \geq \|\gamma^{nr} - \mathbf{\Gamma}(\rho^{\otimes n})\|_1 \geq \|\mathcal{M} \circ \mathcal{E}(\gamma^{nr} - \mathbf{\Gamma}(\rho^{\otimes n}))\|_1$$

and the dimension is $\log |\mathbf{C}_k| = nr$, it follows that

$$\begin{aligned} E_D^\rightarrow(\rho) &\geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\mathbf{\Gamma}} \frac{1}{n} \sup_{\mathcal{M}} I(\mathbf{C}_k)_{ABM} \mathcal{M} \circ \mathcal{E}(\gamma^{nr}) \\ &\quad + \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} (\varepsilon nr + \eta(1 + \frac{\varepsilon}{2}) - \eta(\frac{\varepsilon}{2})) \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\mathbf{\Gamma}} \frac{1}{n} \sup_{\mathcal{M}} I(\mathbf{C}_k)_{ABM} \mathcal{M} \circ \mathcal{E}(\gamma^{nr}) \end{aligned}$$

where the equality holds because the second term converges to zero. From Lemma 4 we know that $\mathcal{E}(\gamma^{nr})$ is Bell private state with uniform probability distribution so, as showed in the proof of Lemma 5, we can write the coherent information and the optimization over measurements as an LOCC-relative entropy. Therefore we find:

$$E_D^\rightarrow(\rho) \geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{\mathbf{\Gamma}} \frac{1}{n} D_C(\gamma^{nr} \| \hat{\gamma}^{nr})$$

because, recall, the shields of $\mathcal{E}(\gamma)$ are all locally equivalent to γ by a phase flip that can be applied independently at $A_k B_k$. At this point we follow the proof of Theorem 11, where it is shown that the repeater protocol can be used for the task of distinguishing the inputs from separable states in the case where Alice and Bob act as a single party (joined together) — see Figure 3 — i.e we can further lower bound with

$$E_D^{\rightarrow}(\rho) \geq \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\Lambda, \Gamma} \frac{1}{n} D(\Lambda(\gamma^{nr}) \parallel \Lambda(\hat{\gamma}^{nr}))$$

where we left implicit that Charlie is now traced out. This is where we use the condition that the key swapping protocol needs to distill irreducible private states, so we can undo the twisting unitary and trace out the shield systems (\mathcal{T}) and be left with a state δ -close to Φ^{nR} . Notice that while $\text{Tr}_C \Lambda(\hat{\gamma}^{nr})$ will still be a separable state, it will not be the key attacked state of γ^{nR} in general, therefore we rename $\sigma_{A:B} := \text{Tr}_C \Lambda(\hat{\gamma}^{nr})$. Then:

$$\begin{aligned} E_D^{\rightarrow}(\rho) &\geq \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\Lambda, \Gamma} \frac{1}{n} D(\mathcal{T} \circ \Lambda(\gamma^{nr}) \parallel \mathcal{T}(\sigma_{A:B})) \\ &\geq \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\Lambda, \Gamma} \frac{1}{n} D(\mathcal{T} \circ \Lambda(\gamma^{nr}) \parallel \mathcal{T}(A:B)) \end{aligned}$$

Now we asymptotic continuity of the relative entropy:

$$\begin{aligned} E_D^{\rightarrow}(\rho) &\geq \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\Lambda, \Gamma} \frac{1}{n} D(\Phi^{nR} \parallel \mathcal{T}(A:B)) \\ &\quad - \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\Lambda, \Gamma} \frac{1}{n} 2\varepsilon \log \frac{2^{2nR+2}}{\varepsilon} \\ &\geq \lim_{\varepsilon, \delta \rightarrow 0} \limsup_{n \rightarrow \infty} R - \lim_{\varepsilon, \delta \rightarrow 0} 4\varepsilon R. \end{aligned}$$

Taking the final limits gives:

$$E_D^{\rightarrow}(\rho) \geq R_D^{\rightarrow}(\rho, \tilde{\rho}). \quad \square$$

Appendix G: SINGLE-COPY KEY REPEATER

We consider now yet another variation of the repeater: the *single copy* key repeater rate R_D^{sc} . Instead of letting Charlie act jointly on arbitrary many copies, we restrict him to act only on a single copy of the states. This should model, for example, memory-less repeater stations that perform their operations fast and do not allow for distillation. Alice and Bob then proceed to distill key as usual with the outcome of the single copy protocol with Charlie. The definition is then as follows.

Definition 33 (Single-copy R_D [26]).

$$R_D^{sc}(\rho, \rho') := \sup_{\Lambda \in \text{LOCC}_{A:C:B}} K_D(\text{Tr}_C \Lambda(\rho \otimes \rho'))$$

where the supremum is taken over all tripartite LOCC protocols of Alice, Charlie and Bob.

It is possible to prove an upper bound in terms of a single copy relative entropy measure. We follow the proof of a similar upper bound that can be found in [26] for the general key repeater rate, the main difference is that in this case there is no regularization, which would make the bound intractable for our purposes. We will later combine this bound with Corollary 6 to express it in terms of the distillable entanglement.

Theorem 34. *For all states ρ and ρ' it holds:*

$$\begin{aligned} R_D^{sc}(\rho, \rho') &\leq E_{R, \text{MC}:AB}^{A:CB}(\rho \otimes \rho') \\ R_D^{sc}(\rho, \rho') &\leq E_{R, \text{MC}:AB}^{AC:B}(\rho \otimes \rho') \end{aligned}$$

where

- $E_{R, \text{MC}:AB}^{A:CB}$ and $E_{R, \text{MC}:AB}^{AC:B}$ are relative entropies with $P = \text{SEP}_{A:CB}$ and $P = \text{SEP}_{AC:B}$ respectively, and $\mathbb{L} = \text{LOCC}_{\text{MC}:AB}$;
- $\text{LOCC}_{\text{MC}:AB}$ are all LOCC protocols of $C:AB$ that end with a measurement at Charlie.

Proof. Without loss of generality let $\sigma \in \text{SEP}_{A:CB}$. Furthermore, for any map $\Lambda \in \text{LOCC}_{A:C:B}$ it holds that:

$$\begin{aligned} \text{Tr}_C \Lambda &\in \text{LOCC}_{\text{MC}:AB} \\ \tilde{\sigma} = \text{Tr}_C \Lambda(\sigma) &\in \text{SEP}_{A:B}. \end{aligned}$$

Therefore, for any such σ and Λ we have:

$$\begin{aligned} D_{\text{MC}:AB}(\rho \otimes \rho' \parallel \sigma) &= \sup_{\mathcal{M}_{\text{MC}:AB}} D(\mathcal{M}(\rho \otimes \rho') \parallel \mathcal{M}(\sigma)) \\ &\geq D(\text{Tr}_C \Lambda(\rho \otimes \rho') \parallel \text{Tr}_C \Lambda(\sigma)) \\ &= D(\text{Tr}_C \Lambda(\rho \otimes \rho') \parallel \tilde{\sigma}) \\ &\geq E_R(\text{Tr}_C \Lambda(\rho \otimes \rho')) \\ &\geq K_D(\text{Tr}_C \Lambda(\rho \otimes \rho')) \end{aligned}$$

where we used that E_R is a known upper bound on K_D [20]. Taking the supremum over all Λ 's we find:

$$D_{\text{MC}:AB}(\rho \otimes \rho' \parallel \sigma_{A:CB}) \geq R_D^{sc}(\rho, \rho').$$

Taking the infimum over σ 's we end the proof. \square

Recall now that Corollary 6 generalizes to the two-way case in the following way:

$$\begin{aligned} E_D(\rho) &\geq \sup_{\mathcal{M}_A \in \text{LOCC}_{A:B}} \frac{1}{2^m} \sum_j D(\mathcal{M}(\rho_j) \parallel \mathcal{M}(\hat{\rho})) \\ &= \sup_{\mathcal{M}_{A:B}} \frac{1}{2^m} \sum_j D(\mathcal{M}(\rho_j) \parallel \mathcal{M}(\hat{\rho})) \end{aligned}$$

As a direct application of Corollary 7 to Theorem 34, we have now the following corollary.

Corollary 35. *Let ρ and ρ' be any pair of key correlated states with at least one separable key attacked state. Then:*

$$R_D^{sc}(\rho, \rho') \leq |A_k| \cdot |B_k| \cdot E_D(\rho \otimes \rho').$$

Proof.

$$\begin{aligned}
R_D^{sc}(\rho, \rho') &\leq D_{\mathbb{M}C:AB}(\rho \otimes \rho' \parallel \hat{\rho} \otimes \hat{\rho}') \\
&= \sup_{\mathcal{M}_{\mathbb{M}C:AB}} D(\mathcal{M}(\rho \otimes \rho') \parallel \mathcal{M}(\hat{\rho} \otimes \hat{\rho}')) \\
&\leq \sup_{\mathcal{M}_{\mathbb{M}C:AB}} \sum_{jk} D(\mathcal{M}(\rho_j \otimes \rho'_k) \parallel \mathcal{M}(\hat{\rho} \otimes \hat{\rho}')) \\
&\leq |A_k B_k| \cdot E_D(\rho \otimes \rho') \quad \square
\end{aligned}$$

Appendix H: EXAMPLES

Example 36 (The Swap private states $\gamma_{\mathbb{S}}$ [18]).

This is a class of Bell private states with $m = 1$, for each dimension $d = |A_s| = |B_s|$ it defines:

$$\gamma_{\mathbb{S}} = \frac{1}{2} \left(1 + \frac{1}{d}\right) \phi_+ \otimes \rho_s + \frac{1}{2} \left(1 - \frac{1}{d}\right) \phi_- \otimes \rho_a \quad (\text{H1})$$

for each dimension $d > 1$, where ρ_s and ρ_a are the symmetric and anti-symmetric states in $\mathbb{C}^d \otimes \mathbb{C}^d$ — the extreme Werner states [3]. In private state form, they are defined by:

$$\sigma = \frac{\mathbb{1}}{d^2} \quad T = \mathbb{1}_2 \otimes (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{S})$$

where $\mathbb{S} = \sum_{i,j=0}^{d-1} |ij\rangle\langle ji|$ is the *swap* operator. Notice that the swap is unitary, thus $\mathbb{S}^\dagger \mathbb{S} = \mathbb{S} \mathbb{S}^\dagger = \mathbb{S}^2 = \mathbb{1}$, and hermitian; this gives the following block form:

$$\gamma_{\mathbb{S}} = \frac{1}{2} \frac{1}{d^2} \begin{bmatrix} \mathbb{1} & 0 & 0 & \mathbb{S} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{S} & 0 & 0 & \mathbb{1} \end{bmatrix}$$

This is a strictly irreducible private state so $K_D(\gamma_{\mathbb{S}}) = 1$. By the log-negativity upper bound on distillable entanglement [13], we have

$$E_D(\gamma_{\mathbb{S}}) \leq E_N(\gamma_{\mathbb{S}}) = \log \left(1 + \frac{1}{d}\right)$$

which vanishes for large enough d . \blacktriangle

For the Swap private states we can immediately apply Corollary 12 and find:

Corollary 37.

$$R_D^{\rightarrow}(\gamma_{\mathbb{S}}, \gamma_{\mathbb{S}}) \leq 2 \log \left(1 + \frac{1}{d}\right) \leq \frac{2}{d \ln 2}.$$

Aside from being the first example of an upper bound on key repeater rate for NPT states, Corollary 37 also improves on the single copy key repeater rate upper bound previously known [26]:

$$R_D^{sc}(\gamma_{\mathbb{S}}, \gamma_{\mathbb{S}}) \leq 12 \frac{\ln d}{d \ln 2} + O\left(\frac{1}{d}\right)$$

Any unitary matrix in d dimensions can be used to define a private state [21]. Here we focus only on the special case in [21] that uses unitaries

$$U = \sum_{ij} \frac{1}{\sqrt{d}} u_{ij} |i\rangle\langle j| \quad (\text{H2})$$

such that $|u_{ij}| = 1$. For each such U we then define the following operators, to be used later in the definition of the private states:

$$\begin{aligned}
\mathbb{U} &= \sum_{ij} u_{ij} |ii\rangle\langle jj| \\
\mathbb{U}^\Gamma &= \sum_{ij} u_{ij} |ij\rangle\langle ji|
\end{aligned}$$

where $(\cdot)^\Gamma$ denotes the partial transpose. Notice that \mathbb{U}^Γ is a unitary and $\frac{\mathbb{U}}{\sqrt{d}}$ is unitary in the maximally correlated subspace, therefore

$$\begin{aligned}
\frac{\mathbb{U}}{\sqrt{d}} \frac{\mathbb{U}^\dagger}{\sqrt{d}} &= \frac{\mathbb{U}^\dagger}{\sqrt{d}} \frac{\mathbb{U}}{\sqrt{d}} = \mathbb{1}_c \\
\mathbb{U}^{\Gamma\dagger} \mathbb{U}^\Gamma &= \mathbb{U}^\Gamma \mathbb{U}^{\Gamma\dagger} = \mathbb{1}.
\end{aligned}$$

Example 38 (The Fourier private states $\gamma_{\mathbb{U}^\Gamma}$ [21]).

The class of Fourier private states defines for $m = 1$, for each $d = |A_s| = |B_s|$ and for each U as in Equation (H2):

$$\sigma = \frac{\mathbb{1}}{d^2} \quad T = \mathbb{1}_2 \otimes (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{U}^\Gamma)$$

or in block form:

$$\gamma_{\mathbb{U}^\Gamma} = \frac{1}{2} \frac{1}{d^2} \begin{bmatrix} \mathbb{1} & 0 & 0 & \mathbb{U}^\Gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathbb{U}^{\Gamma\dagger} & 0 & 0 & \mathbb{1} \end{bmatrix}. \quad (\text{H3})$$

Notice that in general these are not Bell private states because \mathbb{U}^Γ is in general not hermitian. U is usually taken to be the discrete Fourier transform, thus the name.

This is also a strictly irreducible private state, thus $K_D(\gamma_{\mathbb{U}^\Gamma}) = 1$, and again we have an upper bound on distillable entanglement via the log-negativity:

$$E_D(\gamma_{\mathbb{U}^\Gamma}) \leq E_N(\gamma_{\mathbb{U}^\Gamma}) = \log \left(1 + \frac{1}{\sqrt{d}}\right).$$

\blacktriangle

Corollary 39.

$$R_D^{\rightarrow}(\gamma_{\mathbb{U}^\Gamma}, \gamma_{\mathbb{U}^\Gamma}) \leq 2 \log \left(1 + \frac{1}{\sqrt{d}}\right) \leq \frac{2}{\sqrt{d} \ln 2}.$$

Example 40 (The Flower private states $\gamma_{\mathbb{U}}$ [21]).

Similarly, the class of Flower private states defines for $m = 1$, for each $d = |A_s| = |B_s|$ and for each unitary U :

$$\sigma = \frac{\mathbb{1}_c}{d} \quad T = \mathbb{1}_2 \otimes (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{U})$$

or in block form:

$$\gamma_U = \frac{1}{2} \frac{1}{d} \begin{bmatrix} \mathbb{1}_c & 0 & 0 & \frac{U}{\sqrt{d}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{U^\dagger}{\sqrt{d}} & 0 & 0 & \mathbb{1}_c \end{bmatrix}. \quad (\text{H4})$$

Again, these are not Bell private states in general. The first such an example was the *flower state* [20], which is obtained when U is tensor products of the Hadamard transform.

They are still strictly irreducible private states, thus $K(\gamma_U) = 1$. However, for the same reason that makes the log-negativity of the Fourier private states small, the log-negativity of the Flower private states becomes large and thus it cannot be used to find a meaningful bound on the distillable entanglement:

$$E_N(\gamma_U) = \log(1 + \sqrt{d}).$$

Indeed, this can be far from the relative entropy of entanglement. Just like for the flower state, $E_R(\gamma_U) = 1$ because the relative entropy of entanglement is non lockable and γ_U becomes separable after measuring either key system in the computational basis [19]. On the other hand, we can actually compute the distillable entanglement explicitly via the hashing bound $H(B) - H(AB)$, because γ_U has support only on the maximally correlated subspace of $A_k B_k A_s B_s$ [23]. Since $\frac{U^\Gamma}{\sqrt{d}}$ is a unitary in the maximally correlated subspace, it is simultaneously diagonalizable with $\mathbb{1}_c$ with the diagonal elements all roots of unity. In short, we find

$$H(A_k B_k A_s B_s)_{\gamma_U} = \log d$$

while the marginals are maximally mixed so

$$H(B_k B_s)_{\gamma_U} = 1 + \log d.$$

Therefore, for the class of Flower private states:

$$E_D(\gamma_U) = R(\gamma_U, \gamma_U) = K(\gamma_U) = E_R(\gamma_U) = 1. \quad (\text{H5})$$

▲

Example 41 (The PPT (noisy) private states ξ_{U^Γ} [26]). These are not exact private states, they are approximate private states that can be made arbitrarily close to the Fourier private states while still being PPT. The class of PPT private states defines (for $m = 1$, for each $d = |A_s| = |B_s|$ and for each U as in Equation (H2)):

$$\xi_{U^\Gamma} = \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(\gamma_{U^\Gamma} + \frac{1}{\sqrt{d}} X_{A_k} \hat{\gamma}_U X_{A_k} \right) \quad (\text{H6})$$

where the function of the local bit flip is to move the key attacked state $\hat{\gamma}_U$ in the orthogonal subspace. Namely,

in block form:

$$\xi_{U^\Gamma} = \frac{1}{2} \frac{1}{1 + \frac{1}{\sqrt{d}}} \begin{bmatrix} \frac{1}{d^2} & 0 & 0 & \frac{U^\Gamma}{d^2} \\ 0 & \frac{1}{\sqrt{d}} \frac{\mathbb{1}_c}{d} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{d}} \frac{\mathbb{1}_c}{d} & 0 \\ \frac{U^{\Gamma\dagger}}{d^2} & 0 & 0 & \frac{1}{d^2} \end{bmatrix}.$$

One can check that this noise is just enough to make them PPT, and that remarkably, the amount of noise needed in the mixture goes to zero for large d . The PPT private states are engineered to become close to the set of separable states after partial transposition. Indeed, since $\mathbb{1}$ and $\mathbb{1}_c$ are PPT invariant, we find

$$\xi_{U^\Gamma}^\Gamma = \frac{1}{2} \frac{1}{1 + \frac{1}{\sqrt{d}}} \begin{bmatrix} \frac{1}{d^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{d}} \frac{\mathbb{1}_c}{d} & \frac{1}{\sqrt{d}} \frac{U}{d\sqrt{d}} & 0 \\ 0 & \frac{1}{\sqrt{d}} \frac{U^\dagger}{d\sqrt{d}} & \frac{1}{\sqrt{d}} \frac{\mathbb{1}_c}{d} & 0 \\ 0 & 0 & 0 & \frac{1}{d^2} \end{bmatrix}.$$

and thus:

$$\xi_{U^\Gamma}^\Gamma = \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(\hat{\gamma}_{U^\Gamma} + \frac{1}{\sqrt{d}} X_{A_k} \gamma_U X_{A_k} \right)$$

which is suddenly mostly a separable key attacked state with a vanishing mixture of a Flower private state. ▲

Until now, the PPT private states were the only example in the literature for which the key repeater rate could be upper bound by a computable quantity. Because these private states are PPT, the distillable entanglement is zero — $E_D(\xi_{U^\Gamma}) = 0$ and ξ_{U^Γ} are not exact private state, so we cannot use Corollary 12 directly anymore. Instead, we need to exploit the monotonicity of the key repeater rate under one-way LOCC operations and the fact that PPT private states are obtained by mixing the Fourier private states with noise via a one-way LOCC operation, thus we find:

Corollary 42.

$$R_D^{\rightarrow}(\xi_{U^\Gamma}, \xi_{U^\Gamma}) \leq 2 \log \left(1 + \frac{1}{\sqrt{d}} \right) \leq \frac{2}{\sqrt{d} \ln 2}.$$

Because Charlie is traced out at the end of the key repeater distillation protocol, the key repeater rate is invariant under transposition of the input on Charlie's systems, thus giving the following upper bound on the key repeater rate [26]:

$$R_D(\rho, \tilde{\rho}) \leq \min\{K_D(\rho^\Gamma), K_D(\tilde{\rho}^\Gamma)\}. \quad (\text{H7})$$

The previous upper bound on $R_D(\xi_{U^\Gamma}, \xi_{U^\Gamma})$ was computed by estimating an upper bound on $K_D(\xi_{U^\Gamma}^\Gamma)$:

$$R_D(\xi_{U^\Gamma}, \xi_{U^\Gamma}) \leq \frac{1}{(\sqrt{d} + 1) \ln 2} (2 \ln(2d) + \ln(\sqrt{d} + 1))$$

However this bound is not optimal; used properly, Equation (H7), yields the following bound, which is still better than Corollary 42 and holds for two-way protocols:

Corollary 43.

$$R_D(\xi_{\mathbb{U}^r}, \xi_{\mathbb{U}^r}) \leq \frac{1}{1 + \sqrt{d}}.$$

Proof. By Equation (H7) and convexity of the relative entropy of entanglement we find:

$$\begin{aligned} R_D(\xi_{\mathbb{U}^r}, \xi_{\mathbb{U}^r}) &\leq K_D(\xi_{\mathbb{U}^r}^\Gamma) \\ &\leq E_R(\xi_{\mathbb{U}^r}^\Gamma) \\ &\leq \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(E_R(\hat{\gamma}_{\mathbb{U}^r}) + \frac{1}{\sqrt{d}} E_R(\gamma_{\mathbb{U}}) \right). \end{aligned}$$

However $\hat{\gamma}_{\mathbb{U}^r}$ is separable and, according to Equation (H5), $E_R(\gamma_{\mathbb{U}}) = 1$. Therefore:

$$R_D(\xi_{\mathbb{U}^r}, \xi_{\mathbb{U}^r}) \leq \frac{1}{\sqrt{d} + 1}. \quad \square$$

Example 44 (The PPT invariant (noisy) private states ξ_Γ [21]). By substituting the key attacked state with the Flower private state in Equation (H6), the expression becomes PPT invariant. Namely, the class of PPT invariant private states defines

$$\xi_\Gamma = \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(\gamma_{\mathbb{U}^r} + \frac{1}{\sqrt{d}} X_{A_k} \gamma_{\mathbb{U}} X_{A_k} \right) = (\xi_\Gamma)^\Gamma. \quad (\text{H8})$$

with block form

$$\xi_\Gamma = \frac{1}{2} \frac{1}{1 + \frac{1}{\sqrt{d}}} \begin{bmatrix} \frac{1}{d^2} & 0 & 0 & \frac{\mathbb{U}^\Gamma}{d^2} \\ 0 & \frac{1}{\sqrt{d}} \frac{1_c}{d} & \frac{1}{\sqrt{d}} \frac{\mathbb{U}}{d\sqrt{d}} & 0 \\ 0 & \frac{1}{\sqrt{d}} \frac{\mathbb{U}^\dagger}{d\sqrt{d}} & \frac{1}{\sqrt{d}} \frac{1_c}{d} & 0 \\ \frac{\mathbb{U}^{\Gamma\dagger}}{d^2} & 0 & 0 & \frac{1}{d^2} \end{bmatrix}$$

which is clearly PPT invariant.

The fact that these private states are PPT invariant makes Equation (H7) useless, but a bound can still be computed combining one-way LOCC monotonicity, Corollary 12 and the Rains bound:

Corollary 45.

$$R_D^\rightarrow(\xi_\Gamma, \xi_\Gamma) \leq \frac{2(1 + \frac{1}{\ln 2})}{1 + \sqrt{d}}.$$

Proof. We introduce the following new states:

$$\begin{aligned} \alpha &= \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(|0\rangle\langle 0| \otimes \gamma_{\mathbb{U}^r} + \frac{1}{\sqrt{d}} |1\rangle\langle 1| \otimes \gamma_{\mathbb{U}} \right) \\ \tilde{\alpha} &= \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(|0\rangle\langle 0| \otimes \xi_{\mathbb{U}^r} + \frac{1}{\sqrt{d}} |1\rangle\langle 1| \otimes \hat{\gamma}_{\mathbb{U}} \right). \end{aligned}$$

Who holds the additional qubit is irrelevant, but by making it part of the shield it is possible to show that α is actually a strictly irreducible private state. Indeed

$$\alpha \propto \begin{bmatrix} |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1}_c d^{\frac{1}{2}} & 0 & 0 & |0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (|0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U})^\dagger & 0 & 0 & |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1}_c d^{\frac{1}{2}} \end{bmatrix}$$

where it holds that

$$\begin{aligned} |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \mathbb{1}_c d^{\frac{1}{2}} &= \\ &= \sqrt{(|0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U})(|0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U})^\dagger} \\ &= \sqrt{(|0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U})^\dagger (|0\rangle\langle 0| \otimes \mathbb{U}^\Gamma + |1\rangle\langle 1| \otimes \mathbb{U})} \end{aligned}$$

proving that α is a private state. Since the left hand side is separable, α is also strictly irreducible. One can obtain ξ_Γ from α via LOCC: use the additional qubit to bit flip $\gamma_{\mathbb{U}}$ but not $\gamma_{\mathbb{U}^r}$ and then trace the qubit. Furthermore, $\tilde{\alpha}$ is clearly PPT, since it is mixture of the PPT states $\xi_{\mathbb{U}^r}$ and $\hat{\gamma}_{\mathbb{U}}$, and it is close to α . Indeed we find:

$$\begin{aligned} D(\alpha \| \tilde{\alpha}) &= \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(D(\gamma_{\mathbb{U}^r} \| \xi_{\mathbb{U}^r}) + \frac{1}{\sqrt{d}} D(\gamma_{\mathbb{U}} \| \hat{\gamma}_{\mathbb{U}}) \right) \\ &= \frac{1}{1 + \frac{1}{\sqrt{d}}} \left(\log \left(1 + \frac{1}{\sqrt{d}} \right) + \frac{1}{\sqrt{d}} E_R(\gamma_{\mathbb{U}}) \right) \\ &\leq \frac{\sqrt{d}}{1 + \sqrt{d}} \left(\frac{1}{\sqrt{d} \ln 2} + \frac{1}{\sqrt{d}} \right) \\ &= \frac{1 + \frac{1}{\ln 2}}{1 + \sqrt{d}}. \end{aligned} \quad (\text{H9})$$

Now, we use one-way LOCC monotonicity of R_D^\rightarrow , Lemma 12, $E_D^\rightarrow(\rho) \leq E_R^{PPT}(\rho)$ [15], the fact that $\tilde{\alpha}$ is PPT and Equation (H9), in this order, to show the claim:

$$\begin{aligned} R_D^\rightarrow(\xi_\Gamma, \xi_\Gamma) &\leq R_D^\rightarrow(\alpha, \alpha) \\ &\leq 2E_D^\rightarrow(\alpha) \\ &\leq 2E_R^{PPT}(\alpha) \\ &\leq 2D(\alpha \| \tilde{\alpha}) \\ &\leq \frac{2(1 + \frac{1}{\ln 2})}{1 + \sqrt{d}} \end{aligned} \quad \square$$

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